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Unrestricted Domain Extensions of Dominant Strategy Implementable Allocation Functions

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February 2018

Abstract It is shown that any one-person dominant strategy implementable allocation function on a restricted domain of types can be extended to the unrestricted domain in such a way that dominant strategy implementability is preserved when utility is quasilinear. A sufficient condition is identified for which this extension is essentially unique.

Keywords dominant strategy incentive compatible; implementation theory; mechanism design

1 Introduction

A mechanism consists of an allocation function and a payment function that respectively determine the alternative that is chosen and the payment that must be made by each individual as a function of their reported types. It is well known that for a dominant strategy incentive compatible mechanism, there is no loss of generality if attention is restricted to a one-person mechanism in which the types of all but one individual are fixed. We show that any one-person dominant strategy implementable allocation function g on a restricted domain of types can be extended to the unrestricted domain in such a way that dominant strategy implementability is preserved

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when utility is quasilinear. We identify a sufficient condition for which this extension is essentially unique in a sense made precise below. Much is known about the properties of dominant strategy implementable allocation functions and their implementing payment functions on an unrestricted domain (see, e.g., Cuff et al, 2012; Vohra, 2011). Because g is the restriction of any of its unrestricted domain extensions, the properties of g's extensions can be used to analyze the properties of g itself, particularly when the extension is essentially unique.

For an arbitrary type space, Rochet (1987) identifies a necessary and sufficient condition for an allocation function to be dominant strategy implementable. Gui et al (2004) show that Rochet's conditions are equivalent to all cycles in the corresponding allocation graph having nonnegative length. The allocation graph is a graph derived from the allocation function whose nodes are the alternatives. Gui et al (2004) also show that the partition of the type space into the sets of types that are assigned the same alternative by the allocation function can be identified using polyhedra known as difference sets that are defined using the lengths of the arcs in the allocation graph.[1] Our arguments draw on the analysis by Edelman and Weymark (2017) of the geometric structure of this partition when the cycle lengths in the allocation graph are all zero. They also draw on an alternative characterization of dominant strategy implementability in terms of node potentials due to Heydenreich et al (2009).

In Section 2, we describe the model. Section 3 introduces allocation graphs and states Rochet's Theorem. Difference sets and the zero 2-cycle condition are considered in Section 4. Node potentials are introduced in Section 5. The existence of an unrestricted domain extension of a dominant strategy implementable allocation function is established in Section 6 and a sufficient condition for this extension to be essentially unique is provided in Section 7. Examples illustrating our results are presented in Section 8. In Section 9, we offer some concluding remarks.[2]

2 Preliminaries

As noted in Section 1, there is no loss of generality in restricting attention to one-person mechanisms. The set of alternatives is $A = \{a_1, \ldots, a_m\}$, where $m \geq 2$. An alternative is sometimes referred to by the integer $i \in M = \{1, \ldots, m\}$ that indexes it. The individual's *type* is a vector $v = (v_1, \ldots, v_m)$ $(= v(a_1), \ldots, v(a_m))$, where $v_i = v(a_i)$ is his valuation of the ith alternative. The *type space* (the set of possible types) is V, where $|V| \geq 2$. The type space is *unrestricted* if $V = \mathbb{R}^m$.

The mechanism designer knows that the individual's type is in V, but does not know its value. He designs a *mechanism* (g, π), where $g\colon V \to A$ is an *allocation function* and $\pi\colon V \to \mathbb{R}$ is a *payment function*. These functions specify the alterna-

[1] The main results in Gui et al (2004) also appear in Vohra (2011).

[2] Further details about the material discussed in Sections 2–5 and 9 may be found in Edelman and Weymark (2017), Heydenreich et al (2009), and Vohra (2011).

tive that is chosen and the individual's payment (subsidy, if negative) as a function of his reported type. The type space V is the *domain* of the mechanism.

The individual's utility is his valuation minus his payment, and so is quasilinear. Formally, given the mechanism (g,π), his *utility* is given by

$$v(g(\tilde{v})) - \pi(\tilde{v}) \tag{1}$$

when v is his true type and $\tilde{v}$ is his reported type. The individual reports a type that maximizes his utility, which need not be his true type.

A mechanism (g,π) is *dominant strategy incentive compatible* if

$$v(g(v)) - \pi(v) \geq v(g(\tilde{v})) - \pi(\tilde{v}), \quad \forall v, \tilde{v} \in V. \tag{2}$$

For such a mechanism, the individual has an incentive to report his true type whatever it is. The allocation function g is *dominant strategy implementable* if there exists a payment function π such that (g,π) is *dominant strategy incentive compatible*. We only consider dominant strategy incentive compatible mechanisms.

Dominant strategy implementability has two implications that allow for some simplification. First, the allocation and payment functions only depend on the valuations of the alternatives that are ever chosen, so we can without loss of generality suppose that g is surjective. Second, payments must be the same for types that are allocated the same alternative, so a payment function that implements the allocation function g can be equivalently described by a function $\rho_g \colon M \to \mathbb{R}$, where $\rho_g(i)$ is the payment if the ith alternative is chosen. That is, using ρ_g, $g(v)$ solves the following affine maximization problem:

$$g(v) = a_i \text{ for some } i \in \arg\max_{i \in M} \{v_i - \rho_g(i)\}, \quad \forall v \in V. \tag{3}$$

The fact that g can be implemented by payments that only depend on the chosen alternative is known as the *taxation principle*.

The ith *alternative preimage* is

$$R_i = \{v \in V | g(v) = a_i\}, \quad \forall i \in M. \tag{4}$$

That is, R_i is the set of types that are assigned the ith alternative by g. By assumption, g is surjective, so each of these sets is nonempty.

3 Allocation Graphs and Rochet's Theorem

The *allocation graph* Γ_g corresponding to g is the complete directed graph whose nodes are the set M viewed as labels for the m alternatives. The *length* (which could be negative) of the directed arc from node i to node j is

$$l_{ij} = \inf_{v \in R_j} [v_j - v_i]. \tag{5}$$

By definition, $l_{ii} = 0$ for all $i \in M$. Provided that g is dominant strategy implementable, all of these lengths are finite. Let

$$\bar{l}_i = \frac{1}{m}\sum_j l_{ji}, \quad \forall i \in M, \tag{6}$$

denote the average length of the arcs in Γ_g that terminate at node i.

For any pair of nodes i and j in Γ_g, a *path* is a sequence of directed arcs connecting i to j and a *k-cycle* is a path from i to i with k arcs, where k is any positive integer. The allocation function g satisfies the *k-cycle nonnegativity condition* if all k-cycles in Γ_g have nonnegative length and it satisfies the *zero k-cycle condition* if all k-cycles in Γ_g have zero length.

For an arbitrary type space, Rochet (1987) identifies a necessary and sufficient condition for an allocation function to be dominant strategy implementable. Theorem 1 provides a statement of of Rochet's Theorem in terms of cycles in the allocation graph Γ_g.

Theorem 1 (Rochet (1987)). *The following conditions for the allocation function $g\colon V \to A$ are equivalent:*

1. *g is dominant strategy implementable.*
2. *For every integer $k \geq 2$, the k-cycle nonnegativity condition is satisfied.*

4 Difference Sets and the Zero 2-Cycle Condition

Our analysis exploits the geometric structure of the partition of the type space V provided by the m alternative preimages. This structure is identified using polyhedra defined on all of $\mathbb{R}^m$. In the following, we let int$S$ denotes the interior of the set S and $\mathbf{1}$ denote the vector whose components are all equal to 1.

For all distinct $i, j \in M$, the *pairwise difference set for* the ordered pair of alternatives (a_i, a_j) is

$$\overline{H}_{ij} = \{v \in \mathbb{R}^m | v_i - v_j \geq l_{ji}\} \tag{7}$$

and its boundary is

$$H_{ij} = \{v \in \mathbb{R}^m | v_i - v_j = l_{ji}\}. \tag{8}$$

Each of these pairwise difference sets is a closed halfspace in $\mathbb{R}^m$. It is convenient to let $H_{ii} = \overline{H}_{ii} = \mathbb{R}^m$. For all $i \in M$, the *difference set for a_i* is the polyhedron

$$P_i = \bigcap_{j=1}^{m} \overline{H}_{ij}. \tag{9}$$

As Theorem 2 demonstrates, except for possibly on its boundary, the intersection of the difference set P_i with the type space V is the set of types that are assigned the ith alternative by g.

Theorem 2 (Gui et al (2004)). *For the allocation function $g\colon V \to A$, for any alternative $a_i \in A$:*

1. *For any type $v \in R_i$, $v \in P_i \cap V$.*
2. *If g satisfies the 2-cycle nonnegativity condition, then for any type $v \in \mathrm{int}P_i \cap V$, $v \in R_i$.*

An implication of Theorem 2 is that if $v \in V$ but $v \notin P_i$, then $g(v) \neq a_i$. If $H_{ij} = H_{ji}$, then P_i and P_j have a facet in common and $l_{ij} + l_{ji} = 0$. Dominant strategy implementation implies that the difference sets for distinct alternatives have no interior points in common. As a consequence, if $H_{ij} \neq H_{ji}$, then $l_{ij} + l_{ji} > 0$.

A further implication of Theorem 2 is that if $g(v) = a_i$ and $v' = v + c \cdot \mathbf{1}$, then $g(v') = a_i$ except possibly when v (and, hence v') is on the boundary of P_i. The latter observation permits us to normalize the type vectors so that their components sum to 0 or, equivalently, that they lie in the subspace $\mathbf{1}^\perp$ of $\mathbb{R}^m$ orthogonal to $\mathbf{1}$.

For all $i \in M$, the *normalized difference set for a_i* is

$$\hat{P}_i = P_i \cap \mathbf{1}^\perp. \tag{10}$$

Theorem 3 shows that this set is a pointed cone with vertex p^i whose jth component is the average length of the arcs in Γ_g that terminate at node i minus the length of the arc that goes from node j to node i.

Theorem 3 (Edelman and Weymark (2017)). *For all $i \in M$, $\hat{P}_i$ is a pointed cone with vertex p^i whose jth component is*

$$p^i_j = \bar{l}_i - l_{ji}, \quad \forall j \in M. \tag{11}$$

If the allocation function g is dominant strategy implementable and all of the 2-cycles in Γ_g have zero length, then all cycles in Γ_g have zero length (see Cuff et al, 2012). The relationship between zero cycle lengths and the vertices of the normalized difference sets is provided in Theorem 4.

Theorem 4 (Edelman and Weymark (2017)). *If the allocation function $g\colon V \to A$ is dominant strategy implementable, then the following conditions are equivalent:*

1. *The vertices $\{p^i\}$ of the normalized difference sets $\{\hat{P}_i\}$ coincide.*
2. *g satisfies the zero 2-cycle condition.*

Restrictions on the type space for which the conditions in Theorem 4 are satisfied when the allocation function is dominant strategy implementable have been identified by Cuff et al (2012) and Edelman and Weymark (2017). For example, they hold if the type space is unrestricted.

Because P_i is a cone, $\hat{P}_i$ is the orthogonal projection of P_i onto $\mathbf{1}^\perp$. The orthogonal projection of the type space V onto $\mathbf{1}^\perp$ (the *projected type space*) is also of interest. This projection is denoted by $\hat{V}$.

5 Implementability and Node Potentials

An alternative characterization of dominant strategy implementability to that provided by Rochet's Theorem can be obtained using node potentials. The function $\rho_g \colon M \to \mathbb{R}$ is a *node potential* for the allocation function $g \colon V \to A$ if

$$\rho_g(j) \leq \rho_g(i) + l_{ij}, \quad \forall i, j \in M. \tag{12}$$

That is, a node potential assigns a scalar to each node in the graph Γ_g in such a way that (12) holds.

The payment function $\pi \colon V \to \mathbb{R}$ *corresponds to the node potential* ρ_g if for all $i \in M$ and all $v \in R_i$, $\pi(v) = \rho_g(i)$. In other words, the payment required by the payment function π for any type $v \in V$ that the allocation function g assigns a_i is the value assigned to the ith node in Γ_g by the node potential ρ_g. Theorem 5 provides a characterization of dominant strategy incentive compatibility in terms of node potentials.

Theorem 5 (Heydenreich et al (2009)). *For the allocation function $g \colon V \to A$ and payment function $\pi \colon V \to \mathbb{R}$, (g, π) is dominant strategy incentive compatible if and only if π corresponds to a node potential $\rho_g \colon M \to \mathbb{R}$.*

The node potential ρ_g thus provides a set of implementing payments for the m alternatives. Using Theorems 3 and 4, Edelman and Weymark (2017) show that when the zero 2-cycle condition is satisfied, the common vertex p of the normalized difference sets are implementing payments. By (11), the payment for the ith alternative is then $\bar{l}_i$ (the average length of the arcs that terminate at node i in the allocation graph Γ_g) because $l_{ii} = 0$ for all $i \in M$.

6 Extending the Domain

The allocation function $g^+ \colon \mathbb{R}^m \to A$ is an *unrestricted domain extension* of the allocation function $g \colon V \to A$ if $g^+(v) = g(v)$ for all $v \in V$. Theorem 6 shows that any dominant strategy allocation function on a restricted type space has an unrestricted domain extension that is also dominant strategy implementable.

Theorem 6. *If the allocation function $g \colon V \to A$ is dominant strategy implementable, then g has a unrestricted domain extension $g^+ \colon \mathbb{R}^m \to A$ that is dominant strategy implementable.*

We give two proofs for Theorem 6 that provide different insights about the nature of the extension. The first proof combines a revealed preference argument with the taxation principle's optimization problem in (3).

Proof (Version 1). Because g is dominant strategy implementable, there exists a payment function $\pi \colon V \to \mathbb{R}$ that implements it. By the taxation principle, this payment function can be written as a function $\rho_g \colon M \to \mathbb{R}$ because types that are assigned the same alternative have the same payment. Let

$$\mathscr{O} = \{(a_i, \rho_g(i)) \mid i \in M\} \tag{13}$$

be the set of all combinations of an alternative and its corresponding payment for the mechanism (g, π).

For $v \in V$, let $g^+(v) = g(v)$. For all $v \in \mathbb{R}^m \setminus V$, let

$$g^+(v) = a_i \text{ for some } i \in \arg\max_{i \in M} \{v(a_i) - \rho_g(i)\}. \tag{14}$$

Because there are a finite number of alternatives, $g^+(v)$ is well defined. Thus, when the type v is not in the domain V, the individual gets to choose any one of the alternatives and pays the amount associated with it in the original mechanism. By construction, when the individual is of type v, he is choosing a combination of an alternative and a payment from $\mathscr{O}$ that is utility maximal for him. As a consequence, because $g^+(v) = g(v)$ for $v \in V$, g^+ is an extension of g that is dominant strategy implementable. □

This proof of Theorem 6 is quite simple and highlights the importance of the taxation principle for the construction of the extension of g. However, it does not exploit the geometric structure that is provided by the difference sets and the lengths in the allocation graph that are used to define them. Our second proof of Theorem 6 does.

Consider any dominant strategy implementable allocation function g and let π be a payment function that implements it. By Theorem 5, π corresponds to some node potential ρ_g. Let

$$l^+_{ij} = \rho_g(j) - \rho_g(i), \quad \forall i, j \in M. \tag{15}$$

The value l^+_{ij} is the increment in the payment required if a_j is chosen instead of a_i by the allocation function g using the payment function π corresponding to the node potential ρ_g. The *node potential allocation graph* Γ^+_g is defined to be the complete directed graph with node set M for which the length of the directed arc from node i to node j is l^+_{ij}.

It follows immediately from (15) that every cycle in Γ^+_g has zero length.

Lemma 1. *If $\rho_g \colon M \to \mathbb{R}$ is a node potential for the dominant strategy implementable allocation function $g \colon V \to A$, then for every integer $k \geq 2$, any k-cycle in the node potential allocation graph Γ^+_g has zero length.*

Lemma 2 shows that the length of any arc in the allocation graph Γ_g is at least as large as the length of the corresponding arc in the node potential allocation graph Γ^+_g and that these arc lengths coincide when an arc is part of a zero length 2-cycle of Γ_g.

Lemma 2. *If $\rho_g \colon M \to \mathbb{R}$ is a node potential for the dominant strategy implementable allocation function $g \colon V \to A$, then for all $i, j \in M$,*

$$l_{ij} \geq l_{ij}^{+}. \tag{16}$$

and for all $i, j \in M$ *for which* $l_{ij} + l_{ji} = 0$,

$$l_{ij}^{+} = l_{ij}. \tag{17}$$

Proof. Because ρ_g is a node potential for g, (16) follows from (12) and (15). Consider any $i, j \in M$ for which $l_{ij} + l_{ji} = 0$. Because $l_{ij} + l_{ji} = 0$ and $l_{ij}^{+} + l_{ji}^{+} = 0$, if $l_{ij} > l_{ij}^{+}$, we would have

$$0 = l_{ij} + l_{ji} > l_{ij}^{+} + l_{ji}^{+} = 0,$$

which is impossible. Hence, because (16) holds, (17) does as well. □

For the allocation function $g\colon V \to A$, the *zero 2-cycle graph* is the graph Γ_g^2 with node set M that has an edge between nodes i and j, denoted $i \sim j$, if and only if $l_{ij} + l_{ji} = 0$. This graph is undirected and only has an edge between two nodes if the length of the 2-cycle formed by the arcs connecting these nodes in Γ_g is zero.

For all $i \in M$, let P_i^{+} be the difference set for a_i defined as in (9) but using the lengths $\{l_{ij}^{+}\}$ instead of the lengths $\{l_{ij}\}$ when defining the analogues of the pairwise difference sets in (7). Also let $\hat{P}_i^{+} \subseteq \mathbf{1}^{\perp}$ be the corresponding normalized difference set for a_i. An implication of Lemma 2 is that for all $i \in M$, $P_i \subseteq P_i^{+}$ and $\hat{P}_i \subseteq \hat{P}_i^{+}$. In moving from P_i to P_i^{+}, any facet of P_i that is defined using an alternative whose node forms a 2-cycle of Γ_g^2 with node i is unchanged, whereas any facet of P_i that is defined using an alternative whose node does not form a 2-cycle of Γ_g^2 with node i is moved parallel so as to increase the size of this difference set. We use these observations in our second proof of Theorem 6.

Proof (Version 2). Because g is dominant strategy implementable, by Theorem 5, there exists a node potential $\rho_g\colon M \to \mathbb{R}$ and a payment function $\pi\colon V \to A$ corresponding to it that implements g. By Lemma 2, $l_{ij}^{+} = l_{ij}$ and $l_{ji}^{+} = l_{ji}$ for any pair of nodes i and j for which $i \sim j$ in the 2-cycle graph Γ_g^2. For any pair of nodes i and j for which $i \not\sim j$, by (16), $l_{ij} > l_{ij}^{+}$ and $l_{ji} > l_{ji}^{+}$. Hence, by the definitions of P_i and P_i^{+},

$$P_i \subseteq P_i^{+}, \quad \forall i \in M. \tag{18}$$

We now show that

$$\cup_{i \in M} P_i^{+} = \mathbb{R}^m. \tag{19}$$

On the contrary, suppose that there exists a $v \in \mathbb{R}^m$ for which $v \notin P_i^{+}$ for any $i \in M$. Using the lengths $\{l_{ij}^{+}\}$ instead of the lengths $\{l_{ij}\}$ in (7) and (9), it then follows that for all $i \in M$, there exists an $i_j \in M$ such that

$$v_i - v_{i_j} < l_{i_j i}^{+}. \tag{20}$$

Because the number of nodes is finite, there exists a k-cycle for some $k \in \{2,\dots,M\}$ in which each arc is the arc from i to i_j for some i. Let E be the set of the arcs in this cycle with the arc that starts at node i denoted by ii_j. By (20),

$$0 = \sum_{ii_j \in E} [v_i - v_{i_j}] < \sum_{ii_j \in E} l^{+}_{ii_j}. \tag{21}$$

By Lemma 1, every cycle in the complete directed graph Γ_g^+ has zero length, which contradicts (21). Hence, (19) holds.

We now construct the allocation function $g^+ \colon \mathbb{R}^m \to A$. For all $v \in V$, we let $g^+(v) = g(v)$ so that g^+ is an unrestricted domain extension of g. By construction, $\mathrm{int}P_i^+ \cap \mathrm{int}P_j^+ = \varnothing$ for all $i, j \in M$. For all $i \in M$, let $g^+(v) = a_i$ for any $v \in \mathrm{int}P_i^+ \setminus V$. For any other $v \in \mathbb{R}^m$, there exists a maximal subset $\mathscr{I} \subseteq M$ for which $v \in \cap_{I \in \mathscr{I}} P_i^+$. For such a v, let $g^+(v) = a_i$ for some $i \in \mathscr{I}$. By construction, the allocation function g^+ satisfies the conditions in Theorem 2 reinterpreted so as to apply to g^+.

By Lemma 1, all cycles in Γ_g^+ have zero length. Hence, by Rochet's Theorem (Theorem 1), g^+ is dominant strategy implementable. □

An implication of Theorem 6 is that Γ_g^+ is the allocation graph for the allocation function g^+. Because all 2-cycles in this graph have zero length and g^+ is dominant strategy implementable, it follows from Theorem 4 that the normalized difference sets $\{\hat{P}_i^+\}$ have a common vertex, which we denote by p^+.

7 Essential Uniqueness of an Unrestricted Domain Extension

Two allocation functions g and g' that have the same domain are *essentially equivalent* if their difference sets are identical. By Theorem 2, both of these functions assign the same alternative to any type in their common domain that is in the interior of any of the difference sets. It is only when v is on the boundaries of two or more difference sets that $g(v)$ and $g'(v)$ can differ. An unrestricted domain extension g^+ of an allocation function g is *essentially unique* if any other unrestricted domain extension of g is essentially equivalent to g^+.

In Theorem 7, we show that a dominant strategy implementable allocation function g has an essentially unique unrestricted domain extension if the zero 2-cycle graph Γ_g^2 is connected. This graph need not have any cycles, but as Lemma 3 establishes, if there are any, they must have zero length. This observation is used to help prove our uniqueness result.

Lemma 3. *If the allocation function $g \colon V \to A$ is dominant strategy implementable, then any cycle of the zero 2-cycle graph Γ_g^2 has zero length.*

Proof. By Lemma 2, for any $i, j \in M$ for which $i \sim j$ in Γ_g^2, $l^+_{ij} = l_{ij}$. Because Γ_g^+ is complete and all of its cycles have zero length, it follows that any cycle of Γ_g^2 must have zero length. □

Theorem 7 demonstrates that connectedness of the zero 2-cycle graph is sufficient for the uniqueness of an unrestricted domain extension.

Theorem 7. *If the allocation function* $g\colon V \to A$ *is dominant strategy implementable and the zero 2-cycle graph* Γ_g^2 *is connected, then* g *has an essentially unique unrestricted domain extension* $g^+\colon \mathbb{R}^m \to A$*.*

Proof. Consider any three nodes $i,j,k \in M$ of Γ_g^2 for which $i \sim j$ and $j \sim k$, but $i \not\sim k$. By Lemma 3, the length of the path from node i to node k via node j is the negative of the reverse path. Adding the arc from node k to node i to the first path results in a cycle. Moreover, there is a unique arc length l_{ki}^* that results in this cycle having zero length. The reverse cycle only has zero length if the arc from node i to node k has length $-l_{ki}^*$. The graph Γ_g^2 is connected, and so by assigning lengths in this way, we have uniquely extended Γ_g^2 to a graph for which all three cycles exist and have zero length. A simple induction argument shows that this way of assigning lengths to arcs that are not in Γ_g^2 uniquely extends Γ_g^2 to a complete graph Γ_g^* all of whose cycles have zero length. Lemmas 1 and 2 and Theorem 6 then imply that Γ_g^* coincides with the node potential allocation graph Γ_g^+. The difference sets for any unrestricted domain extension g^+ of g are uniquely determined by the lengths of the arcs in Γ_g^+. Hence, any unrestricted domain extension of g must have the same difference sets and, therefore, there is an essentially unique unrestricted domain extension of g. □

8 Examples

We provide three examples to illustrate how to construct an unrestricted domain extension of an allocation function whose domain is not all of $\mathbb{R}^m$. Edelman and Weymark (2017) use the allocation functions in the first two examples to illustrate Theorem 4, but they do not consider domain extensions.

In each of our examples, there are three alternatives. When this is the case, $\mathbf{1}^\perp$ is a plane, which facilitates the use of diagrams. In our diagrams, the orientation is chosen so that $\mathbf{1}^\perp$ lies flat in the page. Each of the three normalized difference sets $\hat{P}_1$, $\hat{P}_2$, and $\hat{P}_3$ lies in this plane. These sets are pointed cones whose bounding rays form a 120° angle. Because the allocation function g is surjective, each of the normalized difference sets must have a nonempty intersection with the projected type space $\hat{V}$ and each type in $\hat{V}$ must be in at least one of them.

Example 1. A situation in which the conditions in Theorem 4 are satisfied is illustrated in Figure 1. Each pair of normalized difference sets shares a common facet, and so all 2-cycles (and, hence, all cycles) have zero length. By Theorem 4, this is only possible if $\hat{P}_1$, $\hat{P}_2$, and $\hat{P}_3$ share a common vertex. As we have seen in Section 5, the ith component of this vertex is the average length $\bar{l}_i$ of the arcs in Γ_g that terminate at node i.

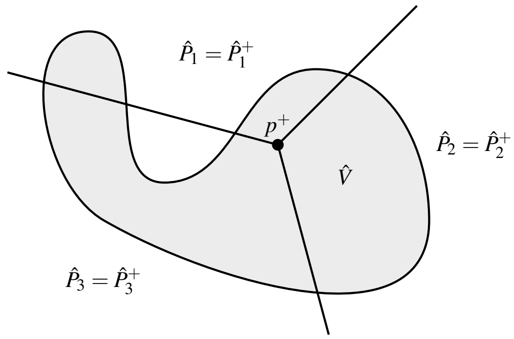

Fig. 1 Illustration of Example 1

To define the allocation function g^+ that extends g to all of $\mathbb{R}^3$, we must, of course, let $g^+(v) = g(v)$ for all $v \in V$. For $v \notin V$, for all $i, j \in M$, g^+ assigns alternative a_i to any $v \in \text{int}P_i$, a_i or a_j to any $v \in P_i \cap P_j$, and a_1, a_2, or a_3 to any $v \in P_1 \cap P_2 \cap P_3$.

In Figure 1, the union of the three normalized difference sets $\{\hat{P}_i\}$ is all of $\mathbf{1}^\perp$ and, hence, the union of the the corresponding difference sets $\{P_i\}$ is all of $\mathbb{R}^m$. As a consequence, for each $i \in M$, the normalized difference set $\hat{P}_i^+$ for g^+ coincides with the corresponding normalized difference set $\hat{P}_i$ for g and, hence, their common vertex p^+ is also the common vertex of $\hat{P}_1$, $\hat{P}_2$, and $\hat{P}_3$.

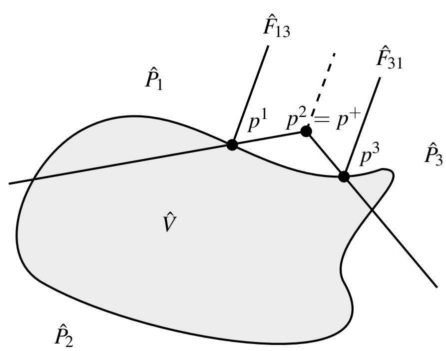

Fig. 2 Illustration of Example 2

Example 2. A situation in which the conditions in Theorem 4 are not satisfied is illustrated in Figure 2. The vertex p^2 of $\hat{P}_2$ lies outside of $\hat{V}$ and differs from the vertices p^1 of $\hat{P}_1$ and p^3 of $\hat{P}_3$. Because the type space V is connected and $m = 3$, there must be at least two zero length 2-cycles (see Edelman and Weymark, 2017; Vohra, 2011). Because the vertices of the normalized difference sets are not all the same, it then follows from Theorem 4 that exactly one of the two cycles has positive length. Here, it is the 2-cycle for a_1 and a_3. This 2-cycle has positive length because $\hat{P}_1$ and $\hat{P}_3$ have no type in common. In contrast, each of the other two pairs of normalized difference sets share a common facet, and so the other 2-cycles have zero length.

There are points in $\mathbb{R}^3$ that are not in any of the normalized difference sets. The allocation function g^+ that extends g to all of $\mathbb{R}^3$ is defined by first constructing difference sets P_1^+, P_2^+, and P_3^+ for which (i) $P_i \subseteq P_i^+$ for all $i \in M$ and (ii) $\cup_{i\in M} P_i = \mathbb{R}^3$. This is done by constructing normalized difference sets $\hat{P}_1^+$, $\hat{P}_2^+$, and $\hat{P}_3^+$ for which (i) $\hat{P}_i \subseteq P_i^+$ for all $i \in M$ and (ii) $\cup_{i\in M} \hat{P}_i = \mathbf{1}^\perp$. The only way to do this is to make p^2 the common vertex of $\hat{P}_1^+$, $\hat{P}_2^+$, and $\hat{P}_3^+$.

By (8), for each $i, j \in M$, $v_i - v_j = l_{ji}$ on the line $H_{ij} \cap \mathbf{1}^\perp$. Hence, any normalized difference set $\hat{P}_i$ has a facet whose slope is the same as one of the facets of $\hat{P}_j$ for $j \neq i$. In Figure 2, $\hat{F}_{13}$ and $\hat{F}_{31}$ are the parallel facets of $\hat{P}_1$ and $\hat{P}_3$, respectively. The sets $\hat{P}_1^+$ and $\hat{P}_3^+$ are obtained from $\hat{P}_1$ and $\hat{P}_3$ by moving these facets so that they coincide with the dashed line in the figure. The set $\hat{P}_2^+$ is set equal to $\hat{P}_2$.[3] The three normalized difference sets constructed in this way have p^2 as their common vertex p^+. For each $i \in M$, $P_i^+ = \{v \in \mathbb{R}^3 \mid v = \tilde{v} + c \cdot \mathbf{1} \text{ for some } \tilde{v} \in \hat{P}_i^+\}$.

The allocation function g^+ that extends g to all of $\mathbb{R}^3$ is now defined as in Example 1. That is, $g^+(v) = g(v)$ for all $v \in V$ and for all other $v \in \mathbb{R}^3$, for all $i, j \in M$, g^+ assigns alternative a_i to any $v \in \text{int} P_i$, a_i or a_j to any $v \in P_i \cap P_j$, and a_1, a_2, or a_3 to any $v \in P_1 \cap P_2 \cap P_3$. All 2-cycles in the corresponding allocation graph have zero length.

In both Examples 1 and 2, that allocation function g has an essentially unique unrestricted domain extension g^+. Moreover, the common vertex p^+ of the normalized difference sets $\hat{P}_1^+$, $\hat{P}_2^+$, and $\hat{P}_3^+$ for g^+ coincides with some of the vertices of the normalized difference sets $\hat{P}_1$, $\hat{P}_2$, and $\hat{P}_3$ for g. In Example 3, the allocation function g does not have an essentially unique unrestricted domain extension. For the extension g^+ considered in this example, p^+ does not coincide with a vertex of any of the normalized difference sets for g.

Example 3. The projected type space $\hat{V}$ and the three normalized difference sets $\hat{P}_1$, $\hat{P}_2$, and $\hat{P}_3$ for the allocation function g are as illustrated in Figure 3. Because $\hat{V}$ is not connected, V is not connected either. Because $l_{12} + l_{21} = 0$, the common vertex p^+ of the three normalized difference sets $\hat{P}_1^+$, $\hat{P}_2^+$, and $\hat{P}_3^+$ for the extension g^+ must lie on the line through p^1 and p^2. It must also lie on a line that is parallel to the upward sloping facets of $\hat{P}_1$ and $\hat{P}_3$ and on a line that is parallel to the downward sloping facets of $\hat{P}_2$ and $\hat{P}_3$. Furthermore, it must lie weakly to the right of $\hat{P}_1$ and weakly to the left of $\hat{P}_3$. It is because these constraints leave some freedom about where to locate p^+ that there is not an essentially unique unrestricted domain extension of g. The exact location of p^+ (subject to these constraints) depends on which payment function is used to implement g or, equivalently, what node potential is used.

[3] In Figures 2 and 3, we do not label these three normalized difference sets. However, they are easily identified by our descriptions of their construction.

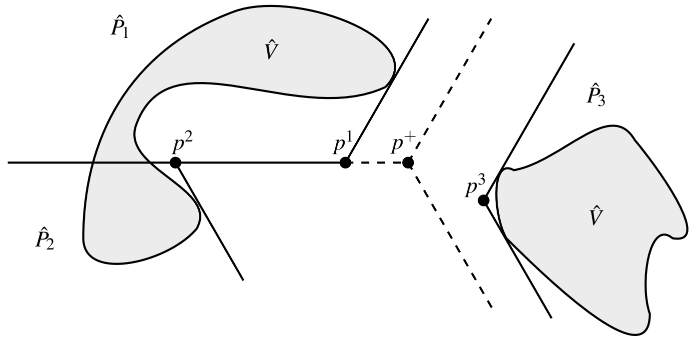

Fig. 3 Illustration of Example 3

The rays that originate at p^+ are the facets of the normalized difference sets for g^+. These sets are used as in Examples 1 and 2 to specify the alternative assigned by g^+ for types that are not in V. All 2-cycles in the allocation graph for g^+ have zero length.

9 Concluding Remarks

A dominant strategy implementable allocation function g satisfies the *revenue equivalence* property if for any two payment functions π and π' that implement it, there exists a scalar c such that

$$\pi'(v) = \pi(v) + c, \quad \forall v \in V. \tag{22}$$

Heydenreich et al (2009) show that revenue equivalence holds if and only if for any two nodes i and j in the allocation graph Γ_g, the length of the shortest path from i to j is the negative of the length of the shortest path from j to i. An implication of this result is that the length of the cycle formed by the shortest paths from node i to j and from node j to i is zero. This cycle need not be a 2-cycle because these paths need not be the direct paths between these two nodes. However, Edelman and Weymark (2017) show that when the zero 2-cycle condition is satisfied, the shortest path between two nodes is the direct path. As a consequence, revenue equivalence is implied by the zero 2-cycle condition when g is dominant strategy implementable. In general, g need not satisfy either the revenue equivalence property or the zero 2-cycle condition. Nevertheless, any unrestricted domain extension of g must satisfy the zero 2-cycle condition because the domain is unrestricted (Cuff et al, 2012) and, hence, it satisfies the revenue equivalence property.

When the zero 2-cycle condition is satisfied by a dominant strategy implementable allocation function g, the normalized difference sets for it and for any unrestricted domain extension are the same. As we have seen, their common vertex p is a set of implementing payments for the alternatives and, hence, the payment function π that corresponds to it is an implementing payment function (as a function of the type). Because this is a situation in which revenue equivalence holds, the set of all payment functions that implement g is the set of all π' that satisfy (22) for some scalar c for the payment function π identified in this way.

It is an open question whether there is a simple way to characterize all of the implementing payment functions for a dominant strategy implementable allocation function g when revenue equivalence does not hold. Such a characterization can be obtained if there exists a simple characterization of the normalized difference sets for all of the unrestricted domain extensions of g when there is not an essentially unique extension. Using the vertices of these normalized difference sets, implementing payments can be identified as is done here for the case in which an unrestricted domain extension is essentially unique.

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