E C O N O M I C S   B U L L E T I N

Antipodality in committee selection

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Abstract

In this paper we compare a minisum and a minimax procedure as suggested by Brams et al. for selecting committees from a set of candidates. Using a general geometric framework as developed by Don Saari for preference aggregation, we show that antipodality of a unique maximin and a unique minisum winner can occur for any number of candidates larger than two.

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1 Introduction

Recently, Brams et al. ([3], see also [2]) have introduced a minimax procedure into the growing literature on committee selection and compared it with a minisum procedure which is equivalent to the application of majority voting on candidates to committee selection. They show that these two rules can lead to the disturbing phenomenon of antipodality, i.e. to opposite oucomes for one and the same preference profile (for the use of antipodality in the comparison of voting rules see Klamler [5] and Ratliff [7]). Antipodality can be seen as evidence for inconsistency of similarly plausible criteria that are incorporated in different rules. This is reminiscent of the inconsistency of similarly plausible properties of aggregation rules in the typical impossibility results in social choice theory. We extend the approach of [2] to a general geometric framework as developed by Saari [8] for preference aggregation and strengthen their result by showing that antipodality of a unique maximin and a unique minisum winner can occur for any number of candidates larger than two.

2 Formal framework and result

For a set of candidates, J, a committee can be represented by a vector of binary evaluations $x = (x^1, x^2, ..., x^{|J|}) \in X \subseteq \{0,1\}^{|J|}$, where $x^j = 1$ means that candidate j belongs to the committee and X denotes the set of all possible committees. Geometrically, X is a subset of the vertices of a $|J|$-dimensional hypercube. (Restrictions on the set X which are ignored here, can guarantee non-emptiness and fixed sizes of committees. On the other hand, the absence of restrictions on the set of binary evaluations establishes a certain analogy to approval voting.[1])

The full preferences of the individuals over committees are derived from their top preferred committees and the Hamming distance. For any two committees $x, y \in X$, the Hamming distance $d(x, y)$ is the number of components (candidates) in which they differ. This means for example that the distance between the committees $x = (1, 0, 0, 0)$ and $y = (0, 1, 0, 1)$ is $d(x, y) = 3$. Hence, an individual with top preferred committee z, considers committee x at least as good as committee y if $d(x, z) \leq d(y, z)$.

A voting profile for this problem of committee selection is a vector $\mathbf{p} = (p_1, p_2, ..., p_{|X|})$ which associates with every binary evaluation $x_k \in X$ the fraction p_k of individuals for which x_k is the top preferred committee, where

[1]This analogy is further explored by Brams et al. in [4].

$\sum_{i=1}^{|X|} p_i = 1$. A major advantage of this geometric representation of profiles lies in the fact that the number of individuals need not be fixed.

A committee selection rule is a mapping that assigns to every profile $\mathbf{p} = (p_1, p_2, ..., p_{|X|})$ a committee $x = f(\mathbf{p})$.[2]

In our framework, for any profile $\mathbf{p} = (p_1, p_2, ..., p_{|X|})$ the outcome of majority voting on candidates ${}^M x = MV(\mathbf{p})$ is defined as follows:

$$\text{For all candidates } j \in J,\ {}^M x^j = 1 \text{ if and only if } \sum_{i=1}^{|X|} p_i x_i^j > 0.5.$$

Following [2], the application of majority voting on candidates to committee selection is equivalent to the distance minimizing rule of minisum. Obviously, for any profile $\mathbf{p} = (p_1, p_2, ..., p_{|X|})$, the *minisum outcome* ${}^M x$ is the committee that, among all $y \in X$, minimizes $\sum_{i=1}^{|X|} p_i d(x_i, y)$. Thus, the minisum committee selection incorporates an efficiency principle in the utilitarian sense of sum-efficiency (see Moulin [6]).

Now, the minimax procedure introduced by Brams et al. [2] selects the committee that minimizes the maximal distance to the individuals' most preferred committees. The major significance of this selection procedure lies in guaranteeing fairness in the well established Rawlsian sense of making the worst off individual as well off as possible. Hence we define for any profile $\mathbf{p} = (p_1, p_2, ..., p_{|X|})$ the *minimax outcome* ${}^S x = MM(\mathbf{p})$ as the outcome that, among all $y \in X$, minimizes $\max d(x_k, y)$ for all k such that $p_k > 0$.

The comparison of the minisum outcome ${}^M x$ and the minimax outcome ${}^S x$ exhibits the disturbing phenomenon of antipodality of the respective outcomes, i.e. there exist profiles for which ${}^M x^j = 0$ if and only if ${}^S x^j = 1$.

Example 1 *The following table states a profile of binary evaluations with $J = 4$ candidates and $N = 19$ voters.*

Looking at candidate 1, we see that she is supported by 10 out of the 19 voters. Hence majority voting gives ${}^M x^1 = 1$. The same applies to candidate 2, whereas candidates 3 and 4 are supported by only 9 voters, therefore ${}^M x^3 = {}^M x^4 = 0$ and thus ${}^M x = (1, 1, 0, 0)$. Now, if we look for the minimax outcome it turns out that there are individuals with every valuation but $(1, 1, 0, 0)$. So for every valuation other than $(0, 0, 1, 1)$ there exists an individual that has exactly the opposite valuation and therefore the minimax distance would be

[2]Possible restrictions on the codomain of the committee selection rule, such as fixed numbers of committee members or the exclusion of certain combinations of members are again ignored here. On the other hand, our framework makes the size of the committee endogeneous.

# voters	evaluation	# voters	evaluation	# voters	evaluation
1	0000	1	1010	2	1110
2	1000	1	1001	2	1101
2	0100	1	0110	1	1011
1	0010	1	0101	1	0111
1	0001	1	0011	1	1111

Table 1: Profile with 19 voters

4. *As there is no individual with binary evaluation* $(1,1,0,0)$ *however, an outcome of* $(0,0,1,1)$ *must minimize the maximal distance to the individual evaluations as such a maximum is* 3. *Hence* ${}^{S}x = (0,0,1,1)$. *Obviously* ${}^{M}x$ *and* ${}^{S}x$ *are antipodal, in the sense that the Hamming distance between these evaluations is maximal because for all candidates* $j \in \{1,2,3,4\}$ ${}^{M}x^{j} = 0$ *if and only if* ${}^{S}x^{j} = 1$.

The above example can be extended to any number of candidates. For the proof of our results we use the geometric framework developed by Saari [8] for the analysis of preference aggregation.

Proposition 1 *For any number of candidates larger than two, there exists a profile of binary evaluations such that the minisum and the minimax outcomes are antipodal.*

The proof proceeds with the help of two lemmas.

Lemma 1 *For any binary evaluation* x_k *there exists a profile* $\mathbf{p}_{-k} = (p_1, p_2, ..., p_k, ..., p_{|X|})$ *such that* $p_k = 0$ *and* $p_i > 0$ *for all* $i \neq k$, *but such that* $x_k = MV(\mathbf{p}_{-k}) = {}^{M}x$.

Proof. *For any binary evaluation* x_k *let* $\{x_a\}^{k}_{a \in A \subset \{1,2,...,|X|\}}$ *denote the set of the* $|J|$ *adjacent vertices of* x_k *and let* $\mathbf{p}^{A}_{-k} = (p_1, p_2, ..., p_k, ..., p_{|X|})$ *denote the profile where* $p_k = 0$ *and* $p_a = \frac{1}{|J|} - \frac{\sum p_l}{|A|}$ *for all* $a \in A$, *where* $p_l = \epsilon$ *is a small positive share for each other evaluation* x_l. *As* x_a *is a neighbor of* x_k *if and only if* $d(x_a, x_k) = 1$, *for each component* $j \in J$, *there are* $|J| - 1$ *evaluations having* $x^{j}_{a} = x^{j}_{k} \in \{0,1\}$. *For* ϵ *being sufficiently small and* $|J| \geq 3$ *this implies that* ${}^{M}x^{j} = 1$ *if and only if* $x^{j}_{k} = 1$ *and hence the lemma is true.* ■

Lemma 2 *For any* $k \in \{1, 2, ..., |X|\}$ *and for any profile* $\mathbf{p}_{-k} = (p_1, p_2, ..., p_k, ..., p_{|X|})$ *such that* $p_k = 0$ *and* $p_i > 0$ *for all* $i \neq k$, $\overline{x}_k = MM(\mathbf{p}_{-k})$, *where* $\overline{x}_k$ *denotes the antipodal evaluation of* x_k, *i.e.* $x_k^j = 1$ *if and only if* $\overline{x}_k^j = 0$. *(The easy proof of this lemma being left to the reader.)*

Proof of Proposition 1. Consider any binary evaluation $x_k \in X$. By lemma 1 it can be obtained as the minisum outcome of some profile $\mathbf{p}_{-k} = (p_1, p_2, ..., p_k, ..., p_{|X|})$. By lemma 2 the minimax outcome of $\mathbf{p}_{-k}$ is $\overline{x}_k$, the antipodal valuation of x_k. ■

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