E c o n o m i c s B u l l e t i n

Local Learning Dynamics

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Abstract

The paper undertakes a detailed characterization of the local dynamic properties of three simple deterministic models involving expectations. The expectations are formed under an adaptive learning process. Allowing for different degrees of learning quality, the analysis reveals the existence of a large variety of possible long term outcomes: in some scenarios, stability and instability are independent of the learning quality; in other circumstances, some minimal requirement on learning efficiency is necessary to attain stability; in some settings, it is even possible that high quality learning may prevent attaining the stable outcome that otherwise is accomplished.

Citation: Gomes, Orlando, (2008) "Local Learning Dynamics." *Economics Bulletin*, Vol. 3, No. 57 pp. 1-15
Submitted: August 15, 2008. **Accepted:** September 16, 2008.
URL: http://economicsbulletin.vanderbilt.edu/2008/volume3/EB-08C60005A.pdf

1 Introduction

If one wants to reasonably address the way private economic agents form expectations about future events, some kind of learning mechanism needs to be introduced. Agents do not possess, from the beginning, all the required information to decide optimally; likewise, typically they lack the capacity to process this information in a completely efficient manner. This simple observation has led economists to resort to more or less sophisticated schemes in which firms and households gather information and process it in order to improve the outcome of their forecasts as time goes by. This literature was pioneered by Marcet and Sargent (1989), in the context of expectations concerning macroeconomic variables, and since then it has been extended in multiple directions.

A strand of this literature highlights the possibility and the desirability of convergence to an asymptotic perfect foresight outcome.[1] Acting as econometricians (i.e., resorting to a least squares algorithm), agents will systematically add new observations to the already collected information, making the impact or gain of new observations to become progressively smaller and eventually converge to zero; in other words, the learning process is fully efficient, meaning that in the long run no more learning will be required, i.e., the perfect foresight equilibrium will be accomplished asymptotically.

An alternative view is the one that suggests that such impact or gain may not fall to zero, meaning that learning persists over time. Resorting to a two-period overlapping generations model, Bullard (1994), Schonhofer (1999) and Tuinstra and Wagener (2007) show that long term outcomes may significantly diverge from a fixed point result. These authors refer to the existence of endogenous fluctuations and chaotic learning equilibria. The rationale under the failure to attain the perfect foresight equilibrium is that a learning mechanism may enclose also a process of memory loss: if agents forget quickly, as they learn, the rational expectations steady state may not be learned.

In synthesis, convergence to the perfect foresight equilibrium under learning offers a logical argument to support the hypothesis of rational expectations in a long run perspective; absence of convergence indicates a form of bounded rationality in which learning and forgetting compete to obtain a less than optimal long term outcome.

By assuming that agents are boundedly rational, thus not being able to learn with full efficiency, some contributions have highlighted the new possibilities that arise in terms of the explanation of macroeconomic phenomena. We make a brief reference to two of these contributions. First, Cellarier (2006) introduces constant gain learning into a framework of optimal growth with infinitely lived agents; this author is able to replicate the existence of business cycles (with properties similar to the ones evidenced by US macroeconomic time series) that under perfect foresight are simply absent. Second, Koulovatianos, Mirman and Santugini (2008) study the effects of introducing learning into the well known Brock-Mirman stochastic growth model; they observe that the new assumption regarding expectations is likely to disturb the optimal consumption and investment paths.

In the sections that follow, we consider three simple generic models involving expectations. Allowing expectations to be formed under the adaptive learning scheme usually found in the literature and taking into consideration the possibility of different long term learning efficiency

[1] This includes, for instance, a significant part of the extensive discussion on monetary policy and learning, surveyed in Evans and Honkapohja (2008).

degrees, we study the local stability properties of each one of the models. In some cases, convergence to the perfect foresight steady state is attained even for relatively low learning efficiency, while in other cases such convergence is ruled out independently of the learning capacity of some representative agent. One observes that slight changes in the model's specification introduce relevant differences in the obtained stability results.

2 The Adaptive Learning Rule

Concerning the formation of expectations, we consider a standard adaptive learning scheme. This is the most widely used learning specification in the economic literature.[2] The framework to present is essentially based on the learning mechanisms proposed by Bullard (1994), Tuinstra and Wagener (2007) and Adam, Marcet and Nicolini (2008).

In an adaptive learning setting, a representative agent resorts to past information in order to form expectations about the future. Assuming that, under perfect foresight, some variable $x_t \in \mathbb{R}$ grows at a constant rate, the perceived law of motion for the evolution of x_t in time will be $E_t x_{t+1} = b_t x_t$, with $b_t - 1$ the growth rate of x_t (if variable x_t is constant in the steady state, then the asymptotic value of b_t will be 1). Because agents have no apriori knowledge about the true value of b_t, this value has to be estimated. A least squares regression is typically used to undertake such estimation.

According to the cited references, the estimation leads to the following dynamic rule characterizing the motion of b_t,

$$b_t = b_{t-1} + \sigma_t \left(\frac{x_{t-1}}{x_{t-2}} - b_{t-1} \right), \quad b_0 \text{ given} \tag{1}$$

Equation (1) reveals that the value of the estimator in t will be equal to its value in the preceding time period plus a term where the estimator is weighted against the last observable change in the value of the endogenous variable (this term measures forecasting errors). Particularly important in this expression is variable σ_t, commonly known as the gain sequence. The gain sequence dictates how past predictions are incorporated into beliefs. Two possibilities are worth noticing: a decreasing gain sequence is such that $\sigma_t \to 0$ as $t \to +\infty$; alternatively, an asymptotic constant gain sequence corresponds to a scenario in which $\sigma_t \to \sigma \in (0, 1)$ as $t \to +\infty$.

The notion of decreasing gain sequence is commonly associated to the idea of a perfect learning process, i.e., to a process that guarantees convergence to the rational expectations equilibrium (to a long run setting in which agents' forecasts are optimal in the limit). The rationale is that agents are able to add new information to the process in each time moment in order to improve the formation of expectations, and therefore, in the steady state all the necessary information acquisition and processing is fulfilled; thus, no more learning is required to form optimal or perfect foresight expectations. A decreasing gain rule would correspond to a difference equation as the following: $\sigma_t = \sigma_{t-1}/(1 + \sigma_{t-1})$, $\sigma_0 \in [0, 1)$ given. This equation has a unique fixed point, $\sigma = 0$, which is stable.

If σ_t does not converge to zero, this might be interpreted as a failure in the learning process. Learning continues to be necessary in the long run, and therefore the representative agent failed

[2] For a comprehensive presentation of the implications of learning in macroeconomic models, we refer the reader to Evans and Honkapohja (2001).

in achieving an optimal forecasting capacity. The more the steady state gain sequence departs from zero, the lower will be the quality of learning, i.e., the lower is the degree of success of the learning process. In Adam, Marcet and Nicolini (2008), it is explicitly stated that as long as the model converges to the rational expectations equilibrium (i.e., as long as σ_t converges to zero), agents' forecasts are, in the limit, optimal. Such an assertion allows to infer that the more the model's outcome diverges from the perfect foresight equilibrium, the more apart the result will be from an efficient or optimal outcome. Constant gain could be given by a rule $\sigma_t = \sigma_{t-1}/(1-\sigma+\sigma_{t-1})$, $\sigma_0 \in [0,1)$ given. This equation has two steady state points; $\sigma = 0$ continues to be a steady state, but now it is an unstable fixed point; the other steady state is stable and equal to $\sigma \in (0,1)$. There is constant gain with a gain sequence σ representing a given level of learning quality.

The analysis that follows concentrates in the study of local stability conditions and, therefore, we avoid addressing explicitly the dynamics of the gain sequence; we consider a constant σ that translates the efficiency of the learning process that culminates in some positive and lower than 1 value for σ.

3 Three Models Involving Expectations

Three different hypotheses concerning the true evolution of variable x_t are considered. A common feature to the three models is that they are deterministic setups all possessing a unique fixed point steady state $\overline{x}$; this steady state is accomplished both when expectations are formed under perfect foresight and under adaptive learning, as long as the underlying system evidences stability. The assumed settings are the following,

I) Variable x_t is a linear function of an array of exogenous variables and of the expected value of the variable in the next period:

$$x_t = f(\mathbf{Y}_t) + \alpha E_t x_{t+1}, \quad \alpha \in \mathbb{R}/\{1\}, f : \mathbb{R}^n \to \mathbb{R} \tag{2}$$

with $\mathbf{Y}_t$ a vector of exogenous variables of dimension n.

II) Variable x_t is a linear function of an array of exogenous variables, of the expected value of the endogenous variable and of the value of x_t in the preceding time period:

$$x_t = f(\mathbf{Y}_t) + \alpha E_t x_{t+1} + \delta x_{t-1}, \quad \alpha \in \mathbb{R}/\{1\}, \delta \in \mathbb{R}, \alpha + \delta \neq 1, f : \mathbb{R}^n \to \mathbb{R} \tag{3}$$

III) Variable x_t is a linear function of an array of exogenous variables and of the previous period expectations about the contemporaneous value of the endogenous variable:

$$x_t = f(\mathbf{Y}_t) + \theta E_{t-1} x_t, \quad \theta \in \mathbb{R}/\{1\}, f : \mathbb{R}^n \to \mathbb{R} \tag{4}$$

The steady state of each one of the equations is straightforward to obtain. In the first model, $\overline{x} := x_t = E_t x_{t+1}$, and thus $\overline{x} = f(\overline{\mathbf{Y}})/(1-\alpha)$; in the second model, $\overline{x} := x_t = E_t x_{t+1} = x_{t-1}$, implying $\overline{x} = f(\overline{\mathbf{Y}})/(1-\alpha-\delta)$; finally, the third model's steady state is defined as $\overline{x} := x_t = E_{t-1} x_t$ and therefore $\overline{x} = f(\overline{\mathbf{Y}})/(1-\theta)$. In every case, $\overline{\mathbf{Y}}$ corresponds to a vector of steady state values of the exogenous variables.

The analysis to undertake corresponds to an inquire about the stability properties of each one of the models in the vicinity of the steady state point. In each case, bifurcation points will separate regions of stability and absence of stability.

3.1 Model I

Rewrite equation (2) in order to the expectations term: $E_t x_{t+1} = x_t/\alpha - f(\mathbf{Y}_t)/\alpha$. If expectations are formed under the specified adaptive learning rule, then $b_t = E_t x_{t+1}/x_t = 1/\alpha - f(\mathbf{Y}_t)/(\alpha x_t)$. Replacing the estimator in the corresponding dynamic equation, one obtains the system,

$$x_t = \frac{f(\mathbf{Y}_t)}{(1-\sigma)f(\mathbf{Y}_t)/x_{t-1} - \sigma(\alpha x_{t-1}/z_{t-1} - 1)};\ z_t = x_{t-1} \tag{5}$$

We recall that, in system (5), σ is the steady state value of the gain sequence, some value between zero and one, that indicates the degree of learning quality. The main result in terms of local stability is the following:

Proposition 1 *In the model where the contemporaneous value of x_t depends on the expected value of the variable for the next period, $\overline{x}$ corresponds to a stable fixed point under conditions:*
1) $\sigma < 2(1-\alpha)/(1-3\alpha)$, as long as $\alpha \in (-\infty, -1) \cup (1, +\infty)$;
2) $\sigma < (1-\alpha)/\alpha$, as long as $\alpha \in (1/2, 1)$;
3) $\forall \sigma \in (0, 1)$, as long as $\alpha \in (-1, 1/2)$.

Proof. See appendix ■

A straightforward corollary is derived from the proposition,

Corollary 2 *In model I, where the current value of the variable is established as depending on the expectations about its next period value, for every $\alpha \notin (-1, 1/2)$ a minimum requirement on learning is needed in order to attain stability, i.e., an upper bound on the gain sequence exists.*

Note the relevance of achieving the steady state under constant gain. It means that although expectations are not formed with full efficiency, the same result (i.e. convergence to $\overline{x}$) is obtained as if asymptotic perfect foresight existed. For some values of parameter α, the quality of learning is irrelevant ($\overline{x}$ is stable independently of the value of σ), while for others a boundary is imposed on the value of σ; the corollary states that in the model in appreciation this is a minimal requirement on the quality of learning that if not attained will imply divergence relatively to the (perfect foresight) steady state.

The bifurcations separating regions of stability and lack of stability are, under condition $\sigma = 2(1-\alpha)/(1-3\alpha)$, a flip bifurcation and, under condition $\sigma = (1-\alpha)/\alpha$, a Neimark-Sacker bifurcation. Thus, if the second stability condition in the proof of proposition 1 is not satisfied then saddle-path stability will hold (only one of the eigenvalues of the Jacobian matrix will locate inside the unit circle); if it is the third stability condition in the proof of proposition 1 that is violated, then instability sets in (the eigenvalues of J will be a pair two complex values with real parts larger than 1 in modulus). Proposition 1 also tells us that for some value of α at most one bifurcation can occur, i.e., flip and Neimark-Sacker bifurcations cannot occur simultaneously in a setting in which we vary the value of σ, maintaining the value of α.

Consider a small example. Assume α as a discount factor (the contemporaneous value of x_t will be given, besides the influence of exogenous variables, by the discounted expected value of the variable in the next period). If, for instance, the discount rate is 5%, then $\alpha = 0.952$. For this value of α, a Neimark-Sacker bifurcation occurs at point $\sigma = 0.0504$, i.e., stability is found only for considerably high levels of learning efficiency, i.e., for values of the gain sequence close to the perfect foresight equilibrium ($\sigma < 0.0504$).

The local dynamics of the model under appreciation is graphically addressed in figure 1.[3] This presents a trace-determinant diagram displaying line segments that refer to different dynamic possibilities for different values of parameter α. Note that for $\sigma = 0$, the system locates over the bifurcation line $1 - Tr(J) + Det(J) = 0$, at the specific point $(Tr(J), Det(J)) = (1, 0)$; the other relevant extreme, $\sigma = 1$, is such that $Tr(J) = Det(J) = \alpha/(1-\alpha)$. A pattern arises: starting at the particular point (1,0), the line reflecting the dynamics is extended to a given point over the main diagonal $Tr(J) = Det(J)$. For some values of α, the stability area (the inverted triangle formed by the bifurcation lines) is not abandoned; in other cases, the flip bifurcation line (negatively sloped bifurcation line) or the Neimark-Sacker bifurcation line (horizontal bifurcation line for $Det(J) = 1$) are crossed and stability is lost. In the figure, five different possibilities of dynamic behavior are drawn, for different values of α. A similar segment of line could be drawn for any other value of the parameter. Such representation confirms the results set forth in proposition 1.

3.2 Model II

The second model involves an additional parameter, δ, and it contains the first specification as a particular case ($\delta = 0$). The procedure to analyze local dynamics is similar to the one used in the previous model. Rearranging equation (3), $E_t x_{t+1} = x_t/\alpha - (\delta/\alpha)x_{t-1} - f(\mathbf{Y}_t)/\alpha$. Recovering the assumed perceived law of motion, $b_t = E_t x_{t+1}/x_t = 1/\alpha - (\delta/\alpha)(x_{t-1}/x_t) - f(\mathbf{Y}_t)/(\alpha x_t)$. This estimator is then replaced in (1) to obtain a two-dimensional system,

$$x_t = \frac{f(\mathbf{Y}_t) + \delta x_{t-1}}{(1-\sigma)(f(\mathbf{Y}_t)/x_{t-1} + \delta z_{t-1}/x_{t-1}) - \sigma(\alpha x_{t-1}/z_{t-1} - 1)};\ z_t = x_{t-1} \tag{6}$$

The local dynamic properties of system (6) are as follows,

Proposition 3 *In the model where the current value of x_t depends on the expected value of the variable for the next period and on the value of the variable in the previous period, the stability of $\overline{x}$ will be characterized by the following conditions,*

1) If $\delta/(1-\alpha) \geq 1$, then stability is absent $\forall \sigma \in (0, 1)$;

2) If $-1 < \delta/(1-\alpha) < 1$, then stability can only be lost through the violation of an upper bound on the gain sequence value. A flip bifurcation occurs as long as $\frac{2(1-\alpha+\delta)}{1-3\alpha+\delta} < 1$ and a Neimark-Sacker bifurcation will be evidenced under $\alpha > 1/2$. The simultaneous verification of conditions $\frac{2(1-\alpha+\delta)}{1-3\alpha+\delta} > 1$ and $\alpha < 1/2$ implies stability $\forall \sigma \in (0, 1)$;

3) If $\delta/(1-\alpha) < -1$, then instability will prevail under $\frac{2(1-\alpha+\delta)}{1-3\alpha+\delta} < 0$ and $\alpha < 1/2$. When $\frac{2(1-\alpha+\delta)}{1-3\alpha+\delta} > 0$ holds, a flip bifurcation occurs and stability will emerge for relatively high values of σ. If, furthermore, $\alpha > 1/2$, an upper bound on the value of σ will also emerge through the occurrence of a Neimark-Sacker bifurcation and above this instability will again prevail.

Proof. See appendix ■

The main difference between models I and II, is that now we allow for new meaningful stability results,

[3] All figures are presented consecutivelly in the end of the paper.

Corollary 4 *The dependence of x_t on its previous period value as well as on the expectations concerning the following period introduces a degree of freedom in the model, allowing to extend the possibilities in terms of local dynamic outcomes. The third situation in proposition 3 indicates that a high quality of learning can be harmful in terms of attaining the stability goal. For values of the gain sequence near the perfect foresight equilibrium, stability is absent, although it can emerge, under certain combinations of parameters and through a flip bifurcation, for lower learning quality standards.*

The results in proposition 3 can be illustrated graphically. In figure 2, three panels are presented, each one corresponding to a different situation in the proposition. Consider, alternatively, $\delta/(1-\alpha) = 2$, $\delta/(1-\alpha) = 1/2$ and $\delta/(1-\alpha) = -2$.

In the first panel, the imposed constraint on parameters implies that $(Tr(J), Det(J)) = (3, 2)$ for $\sigma = 0$; for $\sigma = 1$, depending on the specific value of α, the system will rest in some point over the line $Det(J) = Tr(J) - 2$. In the graphic, some segments of line respecting to possible locations of the dynamics of the system are drawn. One observes that stability is not a feasible result given that the stability inverted triangle is never crossed. In the second panel, condition $\delta/(1-\alpha) = 1/2$ will mean that for $\sigma = 0$ the system locates in point $(Tr(J), Det(J)) = (3/2, 1/2)$, and for $\sigma = 1$ points over line $Det(J) = Tr(J) - 1/2$ are relevant. In this case, it is observable that stability holds for relatively high learning efficiency; however, stability may be lost, for different values of α, through a flip bifurcation or through a Neimark-Sacker bifurcation. Finally, the third panel, drawn for $\delta/(1-\alpha) = -2$, presents line segments starting at $(Tr(J), Det(J)) = (-1, -2)$ and ending in points over the line $Det(J) = Tr(J)+2$. Although for σ close to zero stability is ruled out, it can arise as the result of a flip bifurcation; there is also the possibility (for values of α higher than 1/2 and lower than 1) of a Neimark-Sacker bifurcation implying the loss of stability, in this case for low standards of learning quality. As noted in the proposition, only this last scenario allows to identify an excessive learning quality, that is, high quality learning leading to no convergence towards the perfect foresight steady state.

Consider as an example, that $\alpha = 2/3$. With this value one can quantify the level of the gain sequence needed to attain stability in each of the three scenarios in figure 2. In the first graphic, as mentioned, stability is absent independently of the value of α; in the second panel, stability is guaranteed under an upper limit on σ; in the specific case, $\sigma < 2/3$; finally, the third scenario is such that stability occurs for a value of the gain sequence bounded in the following interval: $\sigma \in (2/5, 1/2)$.

3.3 Model III

We consider, in this third case, that the value of the endogenous variable is formed by weighting last period's expectations about the current value of the variable. This lag in expectations will imply that instead of a two-dimensional system, the system to analyze will have dimension 3 and that the possibilities regarding dynamic results are considerably enlarged relatively to the first model. Rearranging equation (4), $E_{t-1}x_t = x_t/\theta - f(\mathbf{Y}_t)/\theta$. We apply the same perceived law of motion and the same recursive rule for the estimator, in order to obtain the following system of equations,

$$x_{t+1} = (1-\sigma)\left(\frac{x_t^2}{z_t} - f(\mathbf{Y}_t)\frac{x_t}{z_t}\right) + \theta\sigma\frac{x_t z_t}{v_t} + f(\mathbf{Y}_t);\ z_{t+1} = x_t;\ v_{t+1} = z_t \tag{7}$$

Main stability results are synthesized in proposition 5.

Proposition 5 *In the model where the current value of x_t is determined having in consideration last period's expectations concerning the current value of the variable, the stability of $\overline{x}$ will be characterized by the following conditions:*

1) If $\theta \geq 1$, then there is instability $\forall \sigma \in (0,1)$;

2) If $\theta \in [-0.618, 1)$, then there is stability $\forall \sigma \in (0,1)$;

3) If $\theta \in [-1, -0.618)$, then there is stability as long as $\sigma < \sqrt{1/[2(1-\theta)]^2 - 1/\theta} - 1/[2(1-\theta)]$;

4) If $\theta < -1$, then there is stability as long as $\frac{2(\theta+1)}{1+3\theta} < \sigma < \sqrt{1/[2(1-\theta)]^2 - 1/\theta} - 1/[2(1-\theta)]$.

Proof. See appendix ■

Let us compare the possible outcomes in this case with the ones of model I:

Corollary 6 *Just by changing the relevant expectations for the formation of the value of the endogenous variable, we have added new possibilities: instability may exist independently of learning inefficiency; instability may also hold for low and high values of the gain sequence, with a region of stability for intermediate values of the gain variable.*

As in previous models, the graphical illustration of the dynamics using a trace-determinant diagram helps in clarifying the described outcomes. Figure 3 presents three panels. The first one is drawn for $\theta = 2$; the other two take, respectively, $\theta = 1/2$ and $\theta = -2$. One confirms the statements in proposition 5: for $\theta = 2$, instability prevails independently of the value of σ; for $\theta = 1/2$, the line segment characterizing the dynamics of the system falls, in all its extension, inside the unit circle; finally, in the case $\theta = -2$, one observes stability for an intermediate level of the gain sequence. More specifically, stability holds under $\sigma \in (0.4, 0.5598)$. Note that in this case, it is not only the location of the line segment respecting to the dynamics of the system that changes place with variations on the value of the parameter; the stability area also suffers a modification every time θ is changed, since we are drawing the trace-determinant relation, but the stability conditions are also dependent on a third entity that is the sum of the principle minors.

4 Conclusion

Three simple deterministic models involving the determination of the current value of a given variable, when this depends on expectations relating future or present values of the variable, were addressed. Each one of these models possesses a unique steady state point; in the vicinity of such point, a detailed characterization of local stability conditions was undertaken. Local dynamics were studied after replacing the conventional perfect foresight assumption by a mechanism of adaptive learning. Under perfect foresight, the models under consideration are just simple linear difference equations involving trivial dynamics; once one allows for different possibilities regarding the quality or efficiency of the learning process, a large set of possible local dynamic outcomes arises, with local stability depending decisively on the long run value of the gain sequence, which translates the degree of the quality of the learning process. The analysis intends to be a guide for the study of economic models (e.g., relating inflation dynamics) involving expectations formed under adaptive learning and it stresses that slight changes in the model's structure may provoke dramatic changes concerning stability outcomes.

Appendix

Proof of proposition 1. Linearizing (5) in the vicinity of the steady state, one computes the following matricial system,

$$\begin{bmatrix} x_t - \overline{x} \\ z_t - \overline{x} \end{bmatrix} = \begin{bmatrix} 1-\sigma+\alpha\sigma/(1-\alpha) & -\alpha\sigma/(1-\alpha) \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_{t-1} - \overline{x} \\ z_{t-1} - \overline{x} \end{bmatrix}$$

The matricial system evidences that the vector of exogenous variables $\mathbf{Y}_t$ is irrelevant for the study of stability and that this will relate to the elasticity value α and to the gain value σ.

Trace and determinant of the Jacobian matrix are $Tr(J) = 1-\sigma+\alpha\sigma/(1-\alpha)$ and $Det(J) = \alpha\sigma/(1-\alpha)$. The corresponding stability conditions are straightforward to obtain:
$1 - Tr(J) + Det(J) > 0 \Rightarrow \sigma > 0$;
$1 + Tr(J) + Det(J) > 0 \Rightarrow \sigma < 2(1-\alpha)/(1-3\alpha)$;
$1 - Det(J) > 0 \Rightarrow \sigma < (1-\alpha)/\alpha$.

The first stability condition is satisfied for any possible constant gain value $\sigma \in (0, 1)$. The other two conditions will apply for values of α within a certain range. The constraint on the gain sequence implies that the second condition is relevant only for $0 < 2(1-\alpha)/(1-3\alpha) < 1$, and this is equivalent to $\alpha \notin (-1, 1)$ as stated in the proposition; the third condition is meaningful as long as $0 < (1-\alpha)/\alpha < 1$, i.e., if $\alpha \in (1/2, 1)$; for any other value of α, i.e., for $\alpha \in (-1, 1/2)$, all the three conditions are satisfied independently of the level of learning efficiency ■

Proof of proposition 3. In the vicinity of the steady state, system (6) is approximated by,

$$\begin{bmatrix} x_t - \overline{x} \\ z_t - \overline{x} \end{bmatrix} = \begin{bmatrix} 1-\sigma+(\alpha\sigma+\delta)/(1-\alpha) & -(\alpha\sigma+\delta(1-\sigma))/(1-\alpha) \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_{t-1} - \overline{x} \\ z_{t-1} - \overline{x} \end{bmatrix}$$

Trace and determinant of the Jacobian matrix are $Tr(J) = 1-\sigma+(\alpha\sigma+\delta)/(1-\alpha)$ and $Det(J) = (\alpha\sigma+\delta(1-\sigma))/(1-\alpha)$. Stability conditions are straightforward to obtain,
$1 - Tr(J) + Det(J) > 0 \Rightarrow \frac{1-\alpha-\delta}{1-\alpha}\sigma > 0$;
$1 + Tr(J) + Det(J) > 0 \Rightarrow \sigma < \frac{2(1-\alpha+\delta)}{1-3\alpha+\delta}$;
$1 - Det(J) > 1 \Rightarrow \sigma < \frac{1-\alpha-\delta}{\alpha-\delta}$.

To address the stability conditions, let us look at the extreme values of the gain sequence. For $\sigma = 0$, $Tr(J) = 1+\delta/(1-\alpha)$ and $Det(J) = \delta/(1-\alpha)$. Differently from model I, this is not a specific point that is independent of the values of parameters, but it is a collection of points over the bifurcation line $1 - Tr(J) + Det(J) = 0$. The first stability condition is not satisfied if $\delta/(1-\alpha)$ is a non negative quantity, and therefore we can exclude stability for any combination of parameters such that $\delta/(1-\alpha) \geq 1$. Thus, our concern will be only with the values of parameters implying a segment of line translating stability that is placed to the left of $Det(J) = Tr(J) - 1$, and this happens for $\delta/(1-\alpha) < 1$.

For $\sigma = 1$, one has $Tr(J) = (\alpha+\delta)/(1-\alpha)$ and $Det(J) = \alpha/(1-\alpha)$, i.e., $Det(J) = Tr(J) - \delta/(1-\alpha)$. This is a parallel line to $Det(J) = Tr(J) - 1$, and having restricted the analysis to $\delta/(1-\alpha) < 1$, it locates to the left of the bifurcation line. Thus, it is easy to identify two cases with different stability implications. First, if $-1 < \delta/(1-\alpha) < 1$ then the initial point of the line segment giving the local dynamic properties is located inside the unit circle. This line segment crosses the bifurcation lines in the following circumstances: if $\frac{2(1-\alpha+\delta)}{1-3\alpha+\delta} < 1$, a flip bifurcation exists (according to the second stability condition). This condition is equivalent to $\delta \in (-\alpha - 1, 3\alpha - 1)$ for a positive α and $\delta \in (3\alpha - 1, -\alpha - 1)$ for a negative α. If $\frac{1-\alpha-\delta}{\alpha-\delta} < 1$ (third stability condition), then it is guaranteed that a Neimark-Sacker bifurcation occurs; this last condition simplifies to $\alpha > 1/2$.

A second case is the one for which $\delta/(1-\alpha) < -1$; in this scenario, the trace-determinant point for $\sigma = 0$ is outside the unit circle, given that the stability condition $1+Tr(J)+Det(J) > 0$ is violated. A flip bifurcation will exist if $\frac{2(1-\alpha+\delta)}{1-3\alpha+\delta} > 0$, i.e., if $\delta \in (-\infty, \alpha - 1) \cup (3\alpha - 1, \infty)$ for a positive α or $\delta \in (\alpha - 1, \infty) \cup (-\infty, 3\alpha - 1)$ for a negative value of α. A Neimark-Sacker bifurcation will occur once again under $\frac{1-\alpha-\delta}{\alpha-\delta} < 1$, that is, if $\alpha > 1/2$ ∎

Proof of proposition 5. The procedure to arrive to stability conditions is similar to the one used in the previous models. First, we linearize system (7) in the steady state vicinity to obtain it under the matricial form,

$$\begin{bmatrix} x_t - \overline{x} \\ z_t - \overline{x} \\ v_t - \overline{x} \end{bmatrix} = \begin{bmatrix} 1+\theta-\sigma & \theta(2\sigma-1) & -\theta\sigma \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_{t-1} - \overline{x} \\ z_{t-1} - \overline{x} \\ v_{t-1} - \overline{x} \end{bmatrix}$$

For the Jacobian matrix of the problem, we compute trace, $Tr(J) = 1+\theta-\sigma$, determinant, $Det(J) = -\theta\sigma$ and, in this case, also the sum of the principle minors, $\Sigma M(J) = \theta(1-2\sigma)$.

Stability conditions for 3 dimensional discrete time systems are [see Brooks (2004)],

$$\begin{gathered} 1 - Det(J) > 0 \\ 1 - \Sigma M(J) + Tr(J)Det(J) - (Det(J))^2 > 0 \\ 1 - Tr(J) + \Sigma M(J) - Det(J) > 0 \\ 1 + Tr(J) + \Sigma M(J) + Det(J) > 0 \end{gathered}$$

Noticing that $\Sigma M(J) = \theta + 2Det(J)$, the stability conditions may be presented in simplified form taking in consideration just the trace and the determinant,

$$\begin{gathered} Det(J) < 1 \\ Det(J) \in \left(\tfrac{Tr(J)-2}{2} - \sqrt{\left(\tfrac{Tr(J)-2}{2}\right)^2 + 1 - \theta}; \tfrac{Tr(J)-2}{2} + \sqrt{\left(\tfrac{Tr(J)-2}{2}\right)^2 + 1 - \theta} \right) \\ Det(J) > Tr(J) - (1+\theta) \\ Det(J) > -\tfrac{1}{3}\left[Tr(J) + 1 + \theta\right] \end{gathered}$$

We are interested in evaluating the above stability conditions for the relation between the gain sequence and the model's parameter θ. One first straightforward result is given by the third condition, which is equivalent to $\theta < 1$, independently of the value of σ. Thus, if $\theta \geq 1$, stability (i.e., the three eigenvalues of the Jacobian matrix inside the unit circle) will not hold. For the remaining values of θ, the evaluation of the other stability conditions is such that:

1) The stability condition $Det(J) < 1$ is satisfied for any feasible value of σ if $\theta \in [-1, 1)$, while if $\theta < -1$ then condition $\sigma < -1/\theta$ must hold;

2) The second stability condition corresponds to a set in which the determinant of the Jacobian matrix must locate. The lower bound on this set is satisfied $\forall\sigma$; the same is true for the upper bound as long as $\theta \in [-0.618, 1)$. For $\theta < -0.618$, the following relation must hold in order for the stability condition to be a true condition: $\sigma < \sqrt{1/[2(1-\theta)]^2 - 1/\theta} - 1/[2(1-\theta)]$;

3) Finally, the last stability condition holds $\forall\sigma$ for $\theta \in [-1, 1)$, while if $\theta < -1$ then $\sigma > \frac{2(\theta+1)}{1+3\theta}$ guarantees that the stability inequality is verified.

Combining the various relations derived above, one reaches the conditions in the proposition ∎

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Figures

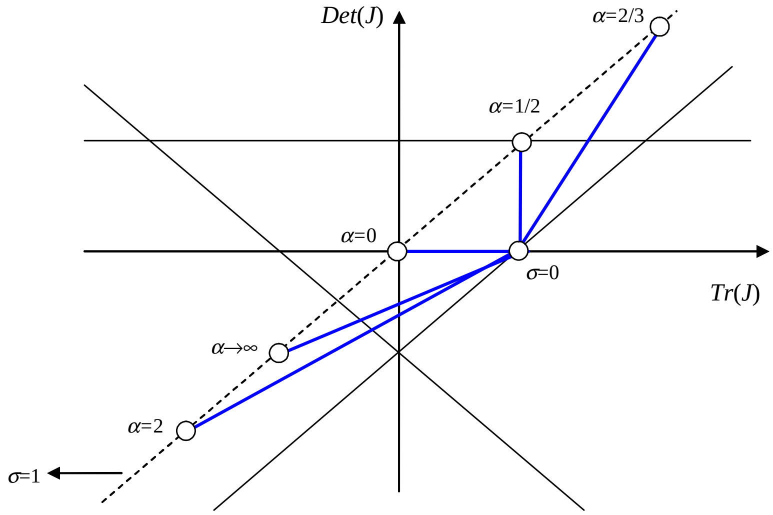

Figure 1: Trace-determinant diagram in model I.

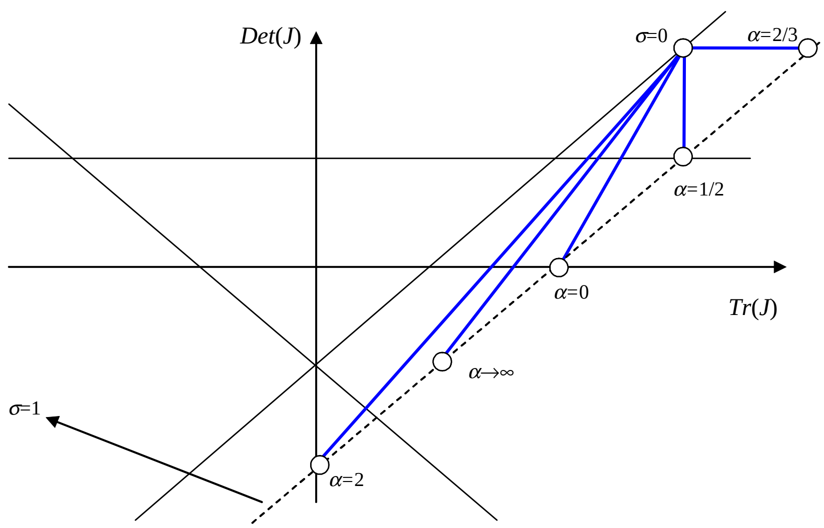

Figure 2a: Trace-determinant diagram in model II ($\delta/(1-\alpha)=2$).

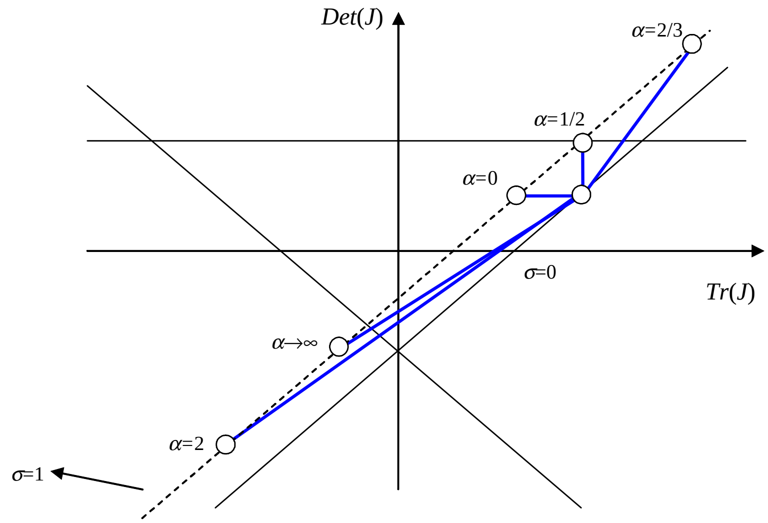

Figure 2b: Trace-determinant diagram in model II ($\delta/(1-\alpha) = 1/2$).

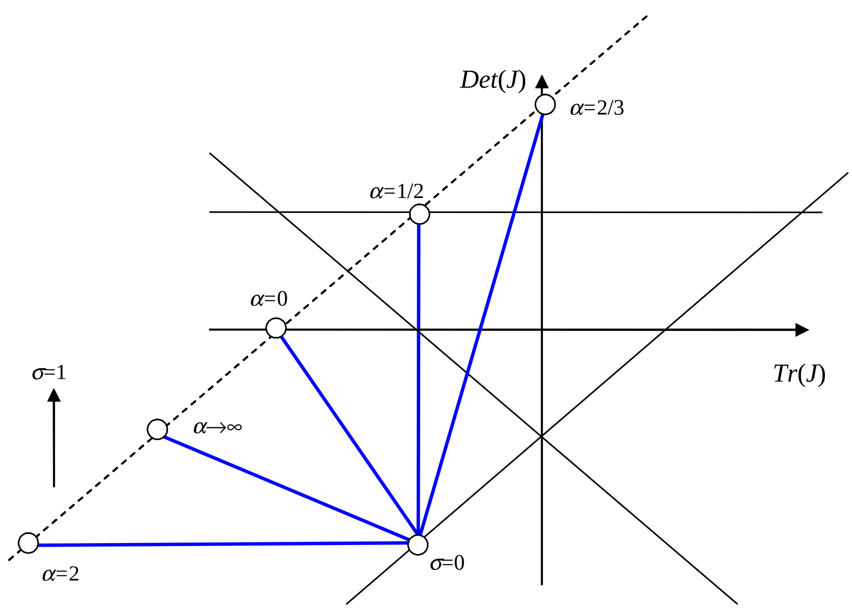

Figure 2c: Trace-determinant diagram in model II ($\delta/(1-\alpha) = -2$).

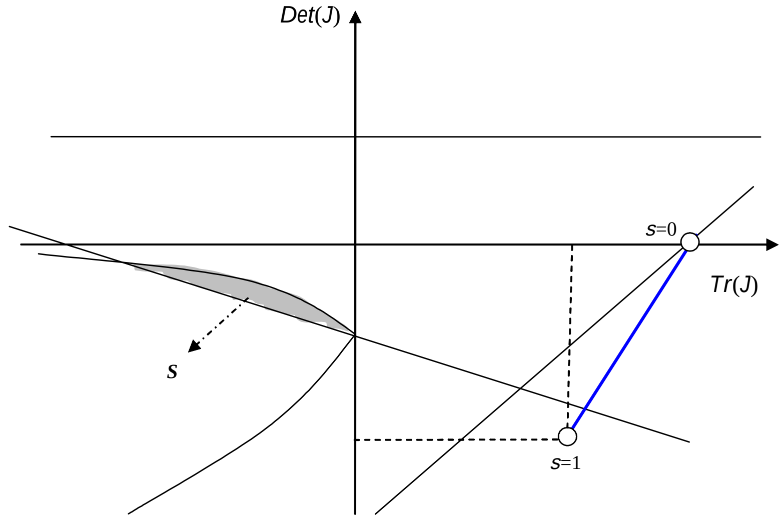

Figure 3a: Trace-determinant diagram in model III ($\theta = 2$).

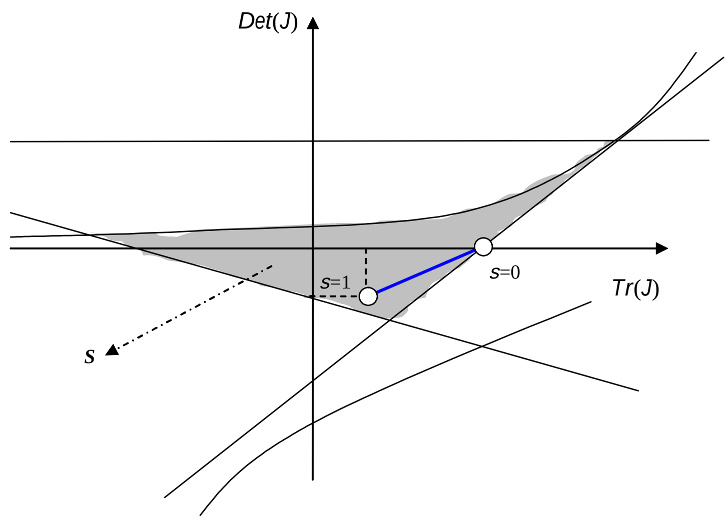

Figure 3b: Trace-determinant diagram in model III ($\theta = 1/2$).

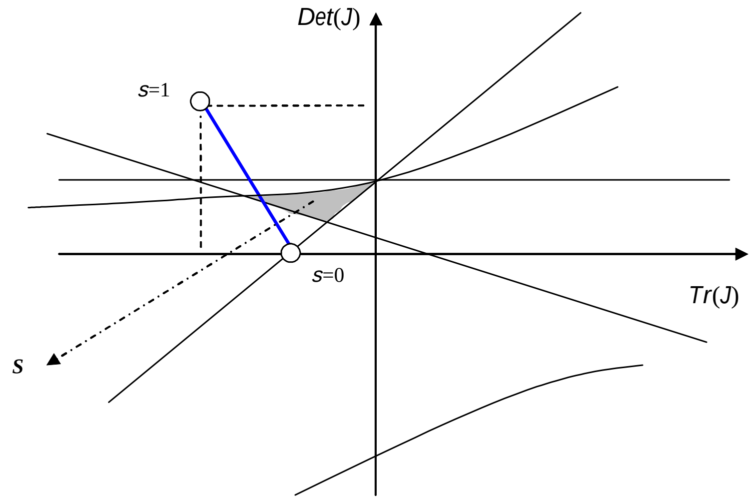

Figure 3c: Trace-determinant diagram in model III ($\theta = -2$).