E C O N O M I C S B U L L E T I N

Estimating the Fractionally Integrated Model with a Break in the Differencing Parameter

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Abstract

This note examines a new problem in the structural-change literature. A fractionally integrated model is assumed to experience a change in the differencing parameter at an unknown time. We develop consistent estimators of the change point and the pre- and post-shift differencing parameters.

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1. Introduction

Fractional integration has been used extensively in modelling long memory processes[1] since the path-breaking work of Granger and Joyeux (1980). Studies on the fractionally integration processes can be categorized into three main areas: (i) Studies on tests for fractionally integration (Cheung, 1993; Wright, 1999; Chen and Deo, 2004; Chong and Hinich, 2007), (ii) Estimation methods for the differencing parameter (Geweke and Porter-Hudak, 1983; Chong, 2000; Mayoral, 2006), and (iii) Empirical estimations of the ARFIMA models (Lo, 1991; Backus and Zin, 1993; Ding et al., 1993; Bailey et al., 1996). For a comprehensive review of the literature on this topic, one is referred to Henry and Zaffaroni (2002) and Robinson (2003). Thus far, most of the studies in the literature assume that the differencing parameter is stable over time. Little attention, however, is paid to the issue of instability of the differencing parameter[2]. In this note, the estimation method of an ARFIMA(0,d,0) process with a break in the differencing parameter will be developed. Our parameters of interest are the pre- and post shift differencing parameters as well as the unknown break point.

[1] A fractionally integrated process is mainly characterized by the differencing parameter which governs the memory property of the process. A positive value of the differencing parameter implies that the process has long memory. It has long been recognized that many macroeconomic time series display long memory property. Recently, Chong (2006) shows that a long memory process can also be obtained by aggregating the AR(1) processes with the autogressive coefficient drawn from a polynomial density.

[2] A related study is Beran and Terrin (1996).

2. The Model

Consider the following model:

$$\begin{aligned}(1-L)^{d_1} y_t &= \varepsilon_t \qquad \text{for } t=1,2,...,k_0, \\ (1-L)^{d_2} y_t &= \varepsilon_t \qquad \text{for } t=k_0+1,k_0+2,...T.\end{aligned} \tag{2.1}$$

The fractional difference operator $(1-L)^d$ is defined by its Maclaurin series

$$(1-L)^d = \sum_{j=0}^{\infty} \frac{\Gamma(j-d)}{\Gamma(-d)\,\Gamma(j+1)} L^j \tag{2.2}$$

where $\Gamma(x)$ is the Euler gamma function defined as

$$\Gamma(x) = \int_0^{\infty} z^{x-1} \exp(-z)\, dz \qquad \text{for } x>0, \tag{2.3}$$

$$\Gamma(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(x+k)\,k!} + \int_1^{\infty} z^{x-1} \exp(-z)\, dz \qquad \text{for } x<0, x\neq -1,-2,-3,... \tag{2.4}$$

The cases for $(d_1, d_2) = (0,1)$ and $(1,0)$ have been explored by Chong (2001). In this note, we will focus on d which takes fractional values.

The following assumptions are made:

$(A1)$ $\varepsilon_t \sim i.i.d.\,(0,\sigma^2)$ $\forall t$, $0<\sigma^2<\infty$ and $E(\varepsilon_t^4)<\infty$;

$(A2)$ $\tau_0 \in [\underline{\tau}, \overline{\tau}] \subset (0,1)$;

If there is no structural change, the h^{th} $(h = 1, 2, 3, ...)$ order autocovariance and autocorrelation of an $ARFIMA(0, d, 0)$ process are given by

$$\gamma_h(d) = \rho_h(d)\,\gamma_0(d) \tag{2.5}$$

and

$$\rho_h(d) = \frac{\Gamma(h+d)\,\Gamma(1-d)}{\Gamma(h+1-d)\,\Gamma(d)} \tag{2.6}$$

respectively, where

$$\gamma_0(d) = \sigma^2 \frac{\Gamma(1-2d)}{\Gamma^2(1-d)}. \tag{2.7}$$

Since there are two regimes in our model, it will be intriguing to uncover the autocovariance and autocorrelation between two observations in different regimes. The following results are new to the literature:

For any $l \leq k_0 < m$, let $h = m - l$, we have

$$\gamma_h(d_1, d_2) = Cov(y_l, y_m) = \sigma^2 \frac{\Gamma(h+d_2)\,\Gamma(1-d_1-d_2)}{\Gamma(h+1-d_1)\,\Gamma(d_2)\,\Gamma(1-d_2)} \tag{2.8}$$

and

$$\begin{aligned} \rho_h(d_1, d_2) &= Corr(y_l, y_m) \\ &= \frac{\gamma_h(d_1, d_2)}{\sqrt{\gamma_0(d_1)}\sqrt{\gamma_0(d_2)}} \\ &= \frac{\Gamma(h+d_2)\,\Gamma(1-d_1)\,\Gamma(1-d_1-d_2)}{\Gamma(h+1-d_1)\,\Gamma(d_2)\,\sqrt{\Gamma(1-2d_1)\,\Gamma(1-2d_2)}}. \end{aligned} \tag{2.9}$$

3. Estimation

To estimate the differencing parameter, the minimum distance estimator of Tieslau et al. (1996) is used. The estimator is defined as

$$\widehat{d} = \underset{d\in(-.5,.25)}{Argmin} S(d), \tag{3.1}$$

where

$S(d) = [\widehat{\boldsymbol{\rho}} - \boldsymbol{\rho}(d)]' C^{-1} [\widehat{\boldsymbol{\rho}} - \boldsymbol{\rho}(d)]$,

$\boldsymbol{\rho}(d)$ is a vector of dimension $(n \times 1)$ with the j^{th} element $\rho_j(d)$, and

$\widehat{\boldsymbol{\rho}}$ is a vector of dimension $(n \times 1)$ with the j^{th} element being the sample autocorrelations defined as

$$\widehat{\rho}_j = \frac{\sum_{t=1}^{T-j} (y_t - \overline{y})(y_{t+j} - \overline{y})}{\sum_{t=1}^{T} (y_t - \overline{y})^2}. \tag{3.2}$$

C is the asymptotic variance covariance matrix of $\widehat{\boldsymbol{\rho}}$ with the $(i,j)^{th}$ element given by

$$c_{i,j} = \sum_{s=1}^{\infty} \left(\rho_{s+i} + \rho_{s-i} - 2\rho_s\rho_i\right)\left(\rho_{s+j} + \rho_{s-j} - 2\rho_s\rho_j\right). \tag{3.3}$$

In this note, we adopt a similar estimator that based on the sample autocorrelation. Let $k = [\tau T]$, where $[\cdot]$ is the greatest integer function. Let

$$\widehat{\tau} = \underset{\tau\in[\underline{\tau},\overline{\tau}]}{Arg\max} M_T(\tau), \tag{3.4}$$

where

$$M_T(\tau) = \left[\widehat{\rho}_1(1,[\tau T]) - \widehat{\rho}_1([\tau T]+1,T)\right]^2, \tag{3.5}$$

$$\widehat{\rho}_1(i,j) = \frac{\sum_{t=i}^{j-1}(y_t-\overline{y})(y_{t+1}-\overline{y})}{\sum_{t=i}^{j-1}(y_t-\overline{y})^2}. \tag{3.6}$$

Thus, if there are two regimes, the break-point estimate is defined to be the time when the difference between the two first-order autocorrelations is maximized. The first-order autocorrelation is used because it has a unique mapping with the differencing parameter. For autocorrelations with order higher than one, the mapping is not unique (Chong, 2000). Thus, it is possible that a change in d may not cause a change in ρ_n for $n > 1$. As a result, one may fail to detect a change in d by inspecting the change in ρ_n.

Simple algebra shows that, for $\tau \leq \tau_0$,

$$M_T(\tau) \xrightarrow{p} \frac{(d_1-d_2)^2}{(1-d_1)^2(1-d_2)^2} h_1^2(\tau) \stackrel{def}{=} M(\tau), \tag{3.7}$$

where

$$h_1(\tau) = \frac{(1-\tau_0)\Gamma(1-2d_2)\Gamma^2(1-d_1)}{(\tau_0-\tau)\Gamma(1-2d_1)\Gamma^2(1-d_2)+(1-\tau_0)\Gamma(1-2d_2)\Gamma^2(1-d_1)}. \tag{3.8}$$

It is worth noting that

$$\frac{\partial}{\partial\tau}M(\tau) = \frac{2(d_1-d_2)^2}{(1-d_1)^2(1-d_2)^2}\frac{\Gamma(1-2d_1)\Gamma^2(1-d_2)}{(1-\tau_0)\Gamma(1-2d_2)\Gamma^2(1-d_1)}h_1^3(\tau) > 0, \tag{3.9}$$

$$\frac{\partial^2}{\partial \tau^2} M(\tau) = \frac{6(d_1-d_2)^2}{(1-d_1)^2(1-d_2)^2}\left[\frac{\Gamma(1-2d_1)\Gamma^2(1-d_2)}{(1-\tau_0)\Gamma(1-2d_2)\Gamma^2(1-d_1)}\right]^2 h_1^4(\tau) > 0. \tag{3.10}$$

For $\tau > \tau_0$,

$$M_T(\tau) \xrightarrow{p} \frac{(d_1-d_2)^2}{(1-d_1)^2(1-d_2)^2} h_2^2(\tau) \overset{def}{=} M(\tau), \tag{3.11}$$

where

$$h_2(\tau) = \frac{\tau_0 \Gamma^2(1-d_2)\Gamma(1-2d_1)}{\tau_0 \Gamma(1-2d_1)\Gamma^2(1-d_2) + (\tau-\tau_0)\Gamma(1-2d_2)\Gamma^2(1-d_1)}. \tag{3.12}$$

Observe that

$$\frac{\partial}{\partial \tau} M(\tau) = -\frac{2(d_1-d_2)^2}{(1-d_1)^2(1-d_2)^2}\frac{\Gamma(1-2d_2)\Gamma^2(1-d_1)}{\tau_0\Gamma^2(1-d_2)\Gamma(1-2d_1)} h_2^3(\tau) > 0, \tag{3.13}$$

$$\frac{\partial^2}{\partial \tau^2} M(\tau) = \frac{6(d_1-d_2)^2}{(1-d_1)^2(1-d_2)^2}\left[\frac{\Gamma(1-2d_2)\Gamma^2(1-d_1)}{\tau_0\Gamma^2(1-d_2)\Gamma(1-2d_1)}\right]^2 h_2^4(\tau) > 0. \tag{3.14}$$

The functions $h_1(\tau)$ and $h_2(\tau)$ take values between zero and one. They equal one when the functions are evaluated at $\tau = \tau_0$. Therefore, from (3.9), (3.10), (3.13) and (3.14), $M_T(\tau)$ converges to a non-stochastic piecewise convex function $M(\tau)$ which is maximized at $\tau = \tau_0$. Under assumptions $(A1)$ to $(A3)$, both the pre- and post-shift processes are stationary with finite variances. Therefore the uniform convergence results of Bai et al. (2007) and Chong (2001, 2003) apply. Hence $\widehat{\tau}$ is a super consistent estimator of τ_0.

After getting the super consistent estimate of the change point, the pre- and post-shift differencing estimators are defined as:

$$\widehat{d}_1 \overset{def}{=} \frac{\widehat{\rho}_1\left(1,\widehat{k}\right)}{1+\widehat{\rho}_1\left(1,\widehat{k}\right)} = \frac{\widehat{\rho}_1\left(1,k_0\right)}{1+\widehat{\rho}_1\left(1,k_0\right)} + o_p\left(1\right) \xrightarrow{p} d_1, \tag{3.15}$$

and

$$\widehat{d}_2 \overset{def}{=} \frac{\widehat{\rho}_1\left(\widehat{k}+1,T\right)}{1+\widehat{\rho}_1\left(\widehat{k}+1,T\right)} = \frac{\widehat{\rho}_1\left(k_0+1,T\right)}{1+\widehat{\rho}_1\left(k_0,+1,T\right)} + o_p\left(1\right) \xrightarrow{p} d_2. \tag{3.16}$$

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