Mobility as movement: A measuring proposal based on transition matrices

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Abstract

In this note we introduce a family of functions that various theoretical results have revealed as useful mobility measures. These functions have enabled us to circumvent an impossibility result obtained by Shorrocks (1978), by adapting one of his axioms to the context of mobility as movement. A particular case belonging to this family is the Bartholomew index, which is widely used in the empirical literature.

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1 Introduction

The vast literature devoted to the study of income inequality is usually formulated in static terms, since it is based on the information supplied by cross-sectional data from the distribution under consideration (see Lambert 1993, or Cowell 1995, for a review). Static analysis provides only a partial view of the distribution examined, however, since it provides no information about the dynamics of the distribution over time, an omission that is particularly important from the social welfare point of view (Friedman 1962). This, along with the increasing availability of longitudinal data sets, explains the growing interest in the measurement of the intra-distribution mobility that can be seen in the literature (see Maasoumi 1998, or Fields and Ok 1999a, for a review).

Intra-distribution mobility can, as a result of its multidimensional nature, be studied from various different approaches. Thus, some authors identify mobility with temporal independence (Shorrocks 1978), while others stress that aspect of mobility that is related to movement *per se* (Fields and Ok 1996, 1999b). Working in line with the latter approach, this note introduces a family of functions that various theoretical results have revealed as useful mobility measures. Specifically, this family of functions has enabled us to circumvent an impossibility result obtained by Shorrocks (1978), by adapting one of his axioms to the context of mobility as movement. A particular case belonging to this family is the Bartholomew index, which is widely used in the empirical literature.

2 Definitions and properties

Let $\mathbb{N}$ be the natural numbers, with $\mathbb{N}^* = (\mathbb{N} \cup \{0\})$. Then, we define $n \in \mathbb{N}$ as the number of individuals in the society. Additionally, let $y_i^{t_0}$ and $y_i^{t_1}$ be the income of individual i at two different points of time, t_0 and t_1, with $y^{t_0} = (y_1^{t_0}, \ldots, y_n^{t_0}) \in \mathbb{R}_+^n$ and $y^{t_1} = (y_1^{t_1}, \ldots, y_n^{t_1}) \in \mathbb{R}_+^n$. Our objective is to measure the intra-distribution mobility between t_0 and t_1. A review of the literature shows that one of the options most commonly used for this purpose involves the construction of transition matrices (Bartholomew 1973, Shorrocks 1978). In order to define the concept of transition matrix, let us now suppose that individuals have been divided into $m \leq n$ non-empty, exhaustive and mutually exclusive classes in ascending order of income level. A transition matrix will be a square matrix $P = [p_{jk}] \in \mathbb{R}_+^{m \times m}$, where p_{jk} denotes the proportion of individuals belonging to class j at t_0 that have shifted to class k at t_1. According to this definition, we have that $\sum_{k=1}^{m} p_{jk} = 1$ for all $j \in \{1, \ldots, m\}$, so that P is a stochastic matrix. Additionally, let Ω be the set of all possible transition matrices. In this literature, a mobility index is defined as a continuous function $M : \Omega \to \mathbb{R}_+$.

It is worth noting that there are different approaches to the study of intra-distribution mobility (Fields and Ok 1999a). The main difference between them being the way in which each one defines those situations characterized by maximum mobility. One alternative, for example, is to identify perfect mobility as the situation in which the probability of moving to any class is independent of that originally occupied (Shorrocks 1978). Note that this implies that $p_{jk} = p_{lk}$ for all $j, k, l \in \{1, \ldots, m\}$. This definition will coincide with maximum mobility only if we identify mobility with the notion of temporal independence. However, as already mentioned in the Introduction, in this note we adopt an alternative approach that highlights the dimension of mobility that is directly related to movement *per se*. In this context, we can intuitively identify perfect mobility with a situation in which all the individuals in each of the m classes move to the class furthest away from their original class. In order to capture this idea, let us consider the set $\vartheta \subset \Omega$, where $P \in \vartheta$ if and only if $p_{jk} = 0$ when $[j \leq \frac{m+1}{2}, k \neq m]$ or $[j \geq \frac{m+1}{2}, k \neq 1]$.

We will now examine a series of basic properties that a mobility measure M based on the information provided by a transition matrix P can reasonably be expected to satisfy.

- *Normalization (NOR): Range* $M(\cdot) = [0, 1]$.
- *Monotonicity (MN):* For all $P, P' \in \Omega$ such that $p_{jk} \geq p'_{jk}$ for all $j \neq k$ and $p_{jk} > p'_{jk}$ for some $j \neq k$, $M(P) > M(P')$.
- *Strong immobility (SIM):* $M(P) = 0$ if and only if $P = I$, where I is the identity matrix.
- *Maximum mobility (MM):* M has a maximum, and if $M(P)$ is a maximum, $P \in \vartheta$.
- *Strong maximum mobility (SMM):* M reaches its maximum in P if and only if $P \in \vartheta$.

NOR, MN and SIM were proposed by Shorrocks (1978). Likewise, MM and SMM are based on an original property imposed by this author, and subsequently adapted to our context of mobility as movement. MM establishes that there is at least one element in ϑ that describes a situation of maximum mobility. In turn, SMM is a strong condition, since it requires that all the transition matrices in ϑ (and no others) represent maximum mobility.

In order to present other properties also considered in our study, we need to denote as P_j the row j of matrix P, that is, $P_j = (p_{j1}, \ldots, p_{jm})$.

- *Independence of irrelevant classes (IIC):* For all $h \leq m$ and for all $P^A, P^B, P^C, P^D \in \Omega$ such that $P_h^A = P_h^C$, $P_h^B = P_h^D$, $P_i^A = P_i^B$ and $P_i^C = P_i^D$ for all $i \neq h$,

$$M(P^A) \geq M(P^B) \Leftrightarrow M(P^C) \geq M(P^D).$$

To clarify the implications of IIC, let us suppose that we have two pairs of transition matrices, (P^A, P^B) and (P^C, P^D), such that in both pairs all rows except h are identical. If row h is equal in matrices P^A and P^C and the same occurs in matrices P^B and P^D, then the comparison in terms of mobility between P^A and P^B should be the same as the ranking between P^C and P^D. Specifically, the idea of IIC is that equal rows play no role in ordinal mobility comparisons. To shed further light on this question, let us consider the following transition matrices:

$$P^A = \begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{bmatrix} \quad P^B = \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}$$
$$P^C = \begin{bmatrix} 0.7 & 0.3 \\ 0 & 1 \end{bmatrix} \quad P^D = \begin{bmatrix} 0.9 & 0.1 \\ 0 & 1 \end{bmatrix}$$

Note that IIC does not establish any comparison between P^A and P^B in terms of their mobility, and the same occurs with P^C and P^D. However, by virtue of this property, we are able to establish that $M(P^A) \geq M(P^B)$ if and only if $M(P^C) \geq M(P^D)$.

- *Symmetry of rows along the main diagonal (SRD):* For all $h \notin \{1, m\}$, for all $\lambda \in [1,\ Min\{h-1,\ m-h\}]$ and for all $P, P' \in \Omega$, if the following conditions hold:

  1. $P_i = P'_i$ for all $i \neq h$,
  2. $p_{hk} = p'_{hk}$ for all $k \notin \{h-\lambda, h+\lambda\}$, and
  3. $p_{h(h-\lambda)} + p_{h(h+\lambda)} = p'_{h(h-\lambda)} + p'_{h(h+\lambda)}$,

  then, $M(P) = M(P')$.

According to SRD, equal degrees of movement away from an initial class should be valued equally by M, irrespective of their direction. To further illustrate the implications of SRD, let us consider the following transition matrices:

$$P^E = \begin{bmatrix} 1 & 0 & 0 \\ 0.2 & 0.5 & 0.3 \\ 0 & 0.3 & 0.7 \end{bmatrix} \quad P^F = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0.3 & 0.7 \end{bmatrix}$$

SRD guarantees that $M(P^E) = M(P^F)$.

3 A family of mobility measures

In the context described in the previous section, lack of movement between classes implies that $P = I$. We can therefore consider the possibility of measuring the mobility from a transition matrix P by calculating the distance between P and I for an appropriate distance function.[1]

Taking this idea as our starting point, let us consider the following family of functions:

Definition 3.1 *A mobility index M belongs to the family of* relative indices of mobility as movement *if there exist $\omega = (\omega_1, \ldots, \omega_m) \in \mathbb{R}_+^m$ with $\sum_{j=1}^{m} \omega_j = 1$, a strictly increasing function $v : \mathbb{N}^* \to \mathbb{R}_+$ and $\alpha \geq 1$ such that for all $P \in \Omega$,*

$$M(P) = MD_{\omega,v,\alpha}(P) = \sum_{j=1}^{m} \omega_j \left[\sum_{k=1}^{m} |p_{jk} - i_{jk}|^\alpha \, v(|j-k|) \right]^{\frac{1}{\alpha}}$$

where i_{jk} is the corresponding element of the identity matrix.

The family of relative indices of mobility as movement includes various measures that differ in parameters ω, v and α. Specifically, parameter ω allows for a different weight, ω_j, to be assigned to each of the m rows. This is not common practice in the literature devoted to the study of intra-distribution mobility using transition matrix data. However, it would appear advisable in empirical analysis to consider possible differences in population or income shares in the various classes. Function v, meanwhile, is included

[1]For example, Dagum (1980), Shorrocks (1982) or Ebert (1984) have used various distance functions within the context of inequality measurement. Up to the present, however, this approach has received very little attention from mobility analysis, save for a few exceptions (Cowell 1985, Fields and Ok 1996).

to assign different weights within each row, according to the degree of movements between classes. Finally, parameter α allows for different distance functions to be considered. We impose that $\alpha \geq 1$, given that in the event of $\alpha < 1$, it would be possible to obtain orderings counterintuitive to the notion of mobility as movement.

Note that a particular case belonging to this family is the mobility measure proposed by Bartholomew (1973), which corresponds with the case of $\alpha = 1$ with v as the identity function.

It is worth mentioning that the indices in this family do not satisfy NOR, given that the range of variation of $MD_{\omega,v,\alpha}(P)$ is not generally limited to the interval $[0,1]$. In fact, it is not possible to establish a predefined upper bound independent of m. Nevertheless, we overcome this problem by normalizing the indices in the following way.

Definition 3.2 *A mobility index M belongs to the family of* normalized relative indices of mobility as movement *if there exist $\omega = (\omega_1, \ldots, \omega_m) \in \mathbb{R}_+^m$ with $\sum_{j=1}^{m} \omega_j = 1$, a strictly increasing function $v : \mathbb{N}^* \to \mathbb{R}_+$ and $\alpha \geq 1$ such that for all $P \in \Omega$,*

$$M(P) = MD_{\omega,v,\alpha}^N(P) = \frac{\sum_{j=1}^{m} \omega_j \left[\sum_{k=1}^{m} \left| p_{jk} - i_{jk} \right|^{\alpha} v(|j-k|) \right]^{\frac{1}{\alpha}}}{\sum_{j=1}^{m} \omega_j [v(0) + Max\{v(j-1), v(m-j)\}]^{\frac{1}{\alpha}}}$$

where i_{jk} is the corresponding element of the identity matrix.

We will now examine the suitability of using the family of normalized relative indices of mobility as movement. For this, let us consider the following result.[2]

Proposition 3.1 *All normalized relative indices of mobility as movement $MD_{\omega,v,\alpha}^N$ satisfy NOR, MN, SIM, MM and IIC.*

It is also worth noting that the family of normalized relative indices of mobility as movement will enable us to obtain a decomposition of observed mobility based on the partition of the population used to define the m classes.

Definition 3.3 *Let $P \in \Omega$, let $MD_{\omega,v,\alpha}^N$ be a normalized relative mobility index and let $j \leq m$. Then, we define the* share of overall mobility attributed to class j in P *according to $MD_{\omega,v,\alpha}^N$ as the following value:*

[2]The proofs of the various results presented in this note are included in the Appendix.

$$C^{j}_{\omega,v,\alpha}(P)=\frac{\omega_j\left[\sum_{k=1}^{m}\left|p_{jk}-i_{jk}\right|^{\alpha}v(|j-k|)\right]^{\frac{1}{\alpha}}}{\sum_{j=1}^{m}\omega_j\left[\sum_{k=1}^{m}\left|p_{jk}-i_{jk}\right|^{\alpha}v(|j-k|)\right]^{\frac{1}{\alpha}}}$$

Note that $C^{j}_{\omega,v,\alpha}(P)$ can be interpreted as the proportion of the decrease that would take place in $MD^{N}_{\omega,v,\alpha}(P)$ assuming there were no movements between classes originating in class j. In fact, it is straightforward that $\sum_{j=1}^{m} C^{j}_{\omega,v,\alpha}(P)=1$.

Although, as we can show in Proposition 3.1, the family of functions $MD^{N}_{\omega,v,\alpha}$ satisfies a group of appealing properties, it contains an infinite number of potential mobility measures with no a priori criteria by which to assess their suitability. To address this problem, we consider the following results.

Proposition 3.2 *$MD^{N}_{\omega,v,\alpha}$ satisfies SMM if and only if $\alpha=1$.*

Proposition 3.3 *$MD^{N}_{\omega,v,\alpha}$ satisfies SRD if and only if $\alpha=1$.*

Remark 3.1 *Propositions 3.1 and 3.2 shows that NOR, MN and SMM are compatible in the context of mobility as movement. In this respect, it is important to note that Shorrocks (1978) proves that this is not the case if we are interested in the notion of mobility related to temporal independence.*

4 Conclusion

In this note we have introduced a family of functions, $MD^{N}_{\omega,v,\alpha}$, which various theoretical results show to be useful mobility measures. These functions enable us to circumvent an impossibility result obtained by Shorrocks (1978), by adapting one of his axioms to the context of mobility as movement. This family also includes the Bartholomew index, which is widely used in the empirical literature.

The most outstanding feature of $MD^{N}_{\omega,v,\alpha}$ is its flexibility, given that, depending on the desired objective, it allows for different weighting schemes to be used for movements between the classes into which the distribution under analysis is divided. It is also possible to decompose $MD^{N}_{\omega,v,\alpha}$ according to the partition used to define the various classes in the population, in order to determine the share of overall mobility attributable to each class. These features suggest that this family of functions may be useful in future empirical work.

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Appendix

Proof of Proposition 3.1

We will show that $MD^N_{\omega,v,\alpha}$ satisfies all the properties of the result, for any possible values of the parameters ω, v, α.

- Normalization (NOR): It is straightforward that $Range\, MD_{\omega,v,\alpha} = [0, \sum_{j=1}^{m} \omega_j \left[v(0) + Max\{v(j-1), v(m-j)\}\right]^{\frac{1}{\alpha}}]$ for all possible parameters ω, v, α. Accordingly, $Range\, MD^N_{\omega,v,\alpha} = [0,1]$.

- Monotonicity (MN): Let there be $P, P' \in \Omega$ such that $p_{jk} \geq p'_{jk}$ for all $j \neq k$ and $p_{jk} > p'_{jk}$ for some $j \neq k$. Let us suppose without loss of generality that there is only one element in which $p_{il} \neq p'_{il}$, with $i \neq l$. We have that

  $MD_{\omega,v,\alpha}(P) - MD_{\omega,v,\alpha}(P') = \omega_i \big\{[(1-p_{ii})^{\alpha} - (1-p'_{ii})^{\alpha}]v(0) + (p^{\alpha}_{il} - {p'}^{\alpha}_{il})v(|l-i|)\big\}$

  Then, given that P and P' are stochastic matrices, we can deduce that $p_{ii} < p'_{ii}$ and, therefore, $(1-p_{ii}) > (1-p'_{ii})$. This, together with $p_{il} > p'_{il}$, implies that $MD_{\omega,v,\alpha}(P) > MD_{\omega,v,\alpha}(P')$ for all possible values of the parameters. Then, $MD^N_{\omega,v,\alpha}(P) > MD^N_{\omega,v,\alpha}(P')$.

- Strong immobility (SIM): To prove the necessary part, if $P = I$, we have that $|p_{jk} - i_{jk}| = 0$, and therefore, $\left[\sum_{k=1}^{m} |p_{jk} - i_{jk}|^{\alpha}\, v(|j-k|)\right]^{\frac{1}{\alpha}} = 0$ for all $j \in \{1, \ldots, m\}$. Then, $MD_{\omega,v,\alpha}(P) = 0$ for all possible values of the parameters. Therefore, $MD^N_{\omega,v,\alpha}(P) = 0$.

  To prove sufficiency, let us consider a matrix P' with $P' \neq I$. Then, $p'_{jk} \geq i_{jk}$ for all $j \neq k$ and $p'_{jk} > i'_{jk}$ for some $j \neq k$. Given that all the indices in this family satisfy MN, $MD^N_{\omega,v,\alpha}(P') > MD^N_{\omega,v,\alpha}(I)$ for all possible values of the parameters.

- Maximum mobility (MM): Let there be $P^0 \in \vartheta$, with P^0 such that $p^0_{jk} \in \{0,1\}$. Then, it can be easily deduced that $MD^N_{\omega,v,\alpha}(P^0) = 1$ for all possible values of the parameters. We will now prove that $MD^N_{\omega,v,\alpha}(P^0) \geq MD^N_{\omega,v,\alpha}(P)$ for all $P \in \Omega$. To this end we consider two separate cases:

  1. $P \in \vartheta$. If $p_{jk} \in \{0,1\}$ for all j,k, clearly, $MD^N_{\omega,v,\alpha}(P) = 1$. Otherwise, there exists a pair (j,k), such that $p_{jk} \notin \{0,1\}$. Since

$P \in \vartheta$, m must be odd, $j = \frac{m+1}{2}$, $k \in \{1, m\}$ and $p_{j1} + p_{jm} = 1$. Therefore, for row j:

$$\left[\sum_{k=1}^{m} |p_{jk} - i_{jk}|^{\alpha} v(|j-k|)\right]^{\frac{1}{\alpha}} = \left[v(0) + (p_{j1}^{\alpha} + p_{jm}^{\alpha}) v(j-1)\right]^{\frac{1}{\alpha}} \leq$$

$$\leq [v(0) + 1^{\alpha} v(j-1)\}]^{\frac{1}{\alpha}}$$

due to the concavity of x^{α} with $\alpha \geq 1$. Then, $MD_{\omega,v,\alpha}(P) \leq \sum_{j=1}^{m} \omega_j \left[v(0) + Max\{v(j-1), v(m-j)\}\right]^{\frac{1}{\alpha}}$. Thus, $MD^{N}_{\omega,v,\alpha}(P) \leq 1$.

2. $P \notin \vartheta$. Then, there must be $p_{jk} \neq 0$ such that

$$k \neq \begin{cases} m \text{ if } j \leq \frac{m+1}{2} \\ \\ 1 \text{ if } j \geq \frac{m+1}{2} \end{cases}$$

Then, we divide the proof into two cases:

 - Case A: $j = k$. Then, for row j, given the concavity of x^{α} with $\alpha \geq 1$, we have the following chain of inequalities:

$$\left[|p_{jj} - 1|^{\alpha} v(0) + \sum_{k \neq j} p_{jk}^{\alpha} v(|j-k|)\right]^{\frac{1}{\alpha}} \leq$$

$$\leq \left[1^{\alpha} v(0) + Max\{v(j-1), v(m-j)\} \sum_{k \neq j} p_{jk}^{\alpha}\right]^{\frac{1}{\alpha}} \leq$$

$$\leq \left[1^{\alpha} v(0) + Max\{v(j-1), v(m-j)\} (\sum_{k \neq j} p_{jk})^{\alpha}\right]^{\frac{1}{\alpha}} <$$

$$< [1^{\alpha} v(0) + Max\{v(j-1), v(m-j)\} 1^{\alpha}]^{\frac{1}{\alpha}} =$$

$$= [v(0) + Max\{v(j-1), v(m-j)\}]^{\frac{1}{\alpha}}.$$

   Then, we have that $MD_{\omega,v,\alpha}(P) < \sum_{j=1}^{m} \omega_j [v(0) + Max\{v(j-1), v(m-j)\}]^{\frac{1}{\alpha}}$. Thus, $MD^{N}_{\omega,v,\alpha}(P) < 1$.

 - Case B: $j \neq k$. An analogous reasoning can be applied:

$$\left[|p_{jj} - 1|^{\alpha} v(0) + \sum_{k \neq j} p_{jk}^{\alpha} v(|j-k|)\right]^{\frac{1}{\alpha}} <$$

$$< \left[1^{\alpha} v(0) + Max\{v(j-1), v(m-j)\} \sum_{k \neq j} p_{jk}^{\alpha}\right]^{\frac{1}{\alpha}} \leq$$

$$\leq \left[1^{\alpha} v(0) + Max\{v(j-1), v(m-j)\} (\sum_{k \neq j} p_{jk})^{\alpha}\right]^{\frac{1}{\alpha}} \leq$$

$\leq [1^{\alpha} v(0) + Max\{v(j-1), v(m-j)\} 1^{\alpha}]^{\frac{1}{\alpha}} =$
$= [v(0) + Max\{v(j-1), v(m-j)\}]^{\frac{1}{\alpha}}$
Again, we have that $MD_{\omega,v,\alpha}(P) < \sum_{j=1}^{m} \omega_j [v(0) + Max\{v(j-1), v(m-j)\}]^{\frac{1}{\alpha}}$. Therefore, $MD^N_{\omega,v,\alpha}(P) < 1$.

- Independence of irrelevant classes (IIC):

  The proof of this property is straightforward, since $MD_{\omega,v,\alpha}(P)$ can also be written as:
  $MD_{\omega,v,\alpha}(P) = \omega_h [\sum_{k=1}^{m} |p_{hk} - i_{hk}|^{\alpha} v(|h-k|)]^{\frac{1}{\alpha}} + \sum_{j \neq h}^{m} \omega_j [\sum_{k=1}^{m} |p_{jk} - i_{jk}|^{\alpha} v(|j-k|)]^{\frac{1}{\alpha}}$

Proof of Proposition 3.2

We have shown in Proposition 3.1 that for all $P \in \vartheta$ such that $p_{jk} \in \{0,1\}$ for all j,k, $MD^N_{\omega,v,\alpha}(P) = 1$ for all possible values of the parameters. We also know that for any $P' \notin \vartheta$, $MD^N_{\omega,v,\alpha}(P') < 1$. Then, we are going to prove that for all $P \in \vartheta$ such that there exists a pair (j,k) with $p_{jk} \notin \{0,1\}$, $MD^N_{\omega,v,\alpha}(P) = 1$ if and only if $\alpha = 1$. In these cases, we have that m must be odd, $j = \frac{m+1}{2}$, $k \in \{1, m\}$ and $p_{j1} + p_{jm} = 1$. If $\alpha = 1$, we have that for row j:

$$\sum_{k=1}^{m} |p_{jk} - i_{jk}| \, v(|j-k|) = v(0) + (p_{j1} + p_{jm}) v(j-1) = v(0) + v(j-1).$$

Accordingly, $MD_{\omega,v,1}(P) = \sum_{j=1}^{m} \omega_j \, [v(0) + Max\{v(j-1), v(m-j)\}]$. Therefore, $MD^N_{\omega,v,1}(P) = 1$.

To prove sufficiency, let us suppose that $\alpha > 1$. Then, for row j:

$$\left[\sum_{k=1}^{m} |p_{jk} - i_{jk}|^{\alpha} \, v(|j-k|)\right]^{\frac{1}{\alpha}} = \left[v(0) + (p_{j1}^{\alpha} + p_{jm}^{\alpha}) v(j-1)\right]^{\frac{1}{\alpha}} <$$
$$< [v(0) + 1^{\alpha} v(j-1)\}]^{\frac{1}{\alpha}}$$

due to the strict concavity of x^{α} with $\alpha > 1$. Accordingly, $MD_{\omega,v,\alpha}(P) < \sum_{j=1}^{m} \omega_j \, [v(0) + Max\{v(j-1), v(m-j)\}]^{\frac{1}{\alpha}}$. Therefore, $MD^N_{\omega,v,\alpha}(P) < 1$.

Proof of Proposition 3.3

It is immediate that $MD^N_{\omega,v,1}$ satisfies SRD. To prove sufficiency, let us consider $P', P'' \in \Omega$ such that:

$$P' = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \quad P'' = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0.5 & 0 & 0.5 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

For all row $j \neq 2$, we have that:

$$\omega_j \left[\sum_{k=1}^{m} \left| p'_{jk} - i_{jk} \right|^{\alpha} v(|j-k|) \right]^{\frac{1}{\alpha}} = \omega_j \left[\sum_{k=1}^{m} \left| p''_{jk} - i_{jk} \right|^{\alpha} v(|j-k|) \right]^{\frac{1}{\alpha}}.$$

And, for row 2, we have that, when $\alpha > 1$:

$$[1^{\alpha} v(1) + v(0)]^{\frac{1}{\alpha}} > [(0.5)^{\alpha} v(1) + v(0) + (0.5)^{\alpha} v(1)]^{\frac{1}{\alpha}}.$$

Therefore, $MD^N_{\omega,v,\alpha}(P') > MD^N_{\omega,v,\alpha}(P'')$, contradicting SRD.