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We apply the mean−standard deviation paradigm to examine a widely used model of the hedging literature. As the hedging model satisfies a scale and location condition the mean−standard deviation technique provides more intuition for the revision of the firm's optimum risk taking when price volatility changes. By introducing risk aversion elasticity we describe the interaction of price risk and optimum hedge. We show that with unit risk aversion elasticity optimum hedge ratio is invariant to changes in price volatilities.

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Optimal hedge ratio and elasticity of risk aversion

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Optimal hedge ratio and elasticity of risk aversion

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We apply the mean-standard deviation paradigm to examine a widely used model of the hedging literature. As the hedging model satisfies a scale and location condition the mean-standard deviation technique provides more intuition for the revision of the firm's optimum risk taking when price volatility changes. By introducing risk aversion elasticity we describe the interaction of price risk and optimum hedge. We show that with unit risk aversion elasticity optimum hedge ratio is invariant to changes in price volatilities.

1. Introduction

Our study analyzes optimal hedging policy of a risk averse firm faced with an exogenous change in price risk. In contrast to the existing literature (see, e.g., Kimball (1990), (1993)) we focus on risk aversion elasticity to characterize the relationship between a change in risk and the optimum hedge ratio. Risk aversion elasticity is defined to be percentage change in risk aversion divided by percentage change in risk. The question how risk affects decision making is an important topic in many fields of economics, insurance and finance.[1] For a (μ, σ)-risk averse firm our note derives a clearcut relationship between changes in risk, the optimum hedge and risk aversion elasticity.

The (μ, σ)-criterion of decision making under uncertainty has experienced a growing attention in very recent contributions.[2] Meyer (1987) shows that if all prospects to be ranked are equal in distribution, except for scale and location, then any expected utility ranking of all prospects can be based on the means and standard deviations of the alternatives' risky outcome. The standard hedging model actually implies that prospects are created through a shifting and scaling process.

[1] See Kimball (1990), Demers and Demers (1991), Briys et al. (1993), Broll et al. (1995), Wong (1996).

[2] See Löffler, (1996), Bar-Shira and Finkelshtain (1999), Ormiston and Schlee (2001), Wagener (2002).

2. Risk aversion elasticity and risk effects

Consider a risk averse firm who owns a certain amount of an asset subject to price risk. At time 1 the firm sells x units at the prevailing spot market price p. This price is random at time 0. However, at time 0, the firm can hedge price risk by taking a short position, i.e. selling contracts H in the futures market. The hedge ratio is then defined as $h = H/x$.

We assume that the risk premium in the futures market is positive and that the firm is (μ, σ)-risk averse. Furthermore, the current futures market price p_f is related to the delivery of one unit of the asset at time 1. There is no basis risk. This allows simplifying the analysis and focusing on particular effects of an increase in risk under (μ, σ)-risk aversion when applying the concept of risk aversion elasticity.

2.1 (μ, σ)-risk aversion

(μ, σ)-risk aversion means that (i) preferences can be represented by a two-parameter function $U(\mu, \sigma)$ defined over mean μ and standard deviation σ of the underlying random variable and (ii) that the function U satisfies the following properties: $\partial U(\mu, \sigma)/\partial \mu = U_\mu > 0$, $\partial^2 U(\mu, \sigma)/\partial \mu^2 = U_{\mu\mu} \leq 0$, $\partial U(\mu, \sigma)/\partial \sigma = U_\sigma < 0$, $\sigma > 0$ and $U_\sigma(\mu, 0) = 0$. We assume that partial derivatives $\partial^2 U(\mu, \sigma)/\partial \sigma^2$ and $\partial^2 U(\mu, \sigma)/\partial \mu \partial \sigma$ exist and that U is a strictly concave function. Indifference curves are convex in (σ, μ)-space as often assumed in the literature.

Given (μ, σ)-risk aversion the hedging decision problem of the firm reads:

$$\max_h U(\mu_{\tilde{w}}, \sigma_{\tilde{w}}),$$

where $\tilde{w} = \tilde{p}(1 - h)x + p_f hx$ denotes uncertain income, with hedge ratio h. We set $\mu_{\tilde{w}} = E(\tilde{w})$ and $\sigma_{\tilde{w}} = \sqrt{E(\tilde{w} - E(\tilde{w}))^2} > 0$.

Before analyzing a change in risk and its effect upon the optimal hedge ratio h, let us introduce risk aversion elasticity. To simplify notation we drop subscript $\tilde{w}$.

Definition: Let $\sigma > 0$. We define the elasticity of risk aversion with regard to the standard deviation as $\varepsilon_{S,\sigma}$ where $S = -U_\sigma/U_\mu$ denotes the risk aversion measure.

2.2 Risk changes and the hedge ratio

We model a change in price risk as follows: $\tilde{p}(\beta) = E\tilde{p} + \beta(\tilde{p} - E\tilde{p})$, where the random variable $\tilde{p}$ has unit standard deviation and $0 < \beta < 1$. Then, increasing β models an increase in price risk. Substituting $\tilde{p}(\beta)$ for the random variable $\tilde{p}$ of the hedging decision problem generates a relationship between the optimal hedge ratio and price risk measured by the standard deviation of $\tilde{p}(\beta)$.

Now we are ready to claim the following

Proposition: Assume backwardation in the forward/futures market. The firm's optimum hedge ratio will increase when price risk increases if and only if risk aversion elasticity is less than unity. With unit risk aversion elasticity price risk changes will not alter optimum hedge ratio.

Proof. Expected income and the standard deviation of income are given by

$$E(\tilde{w}) = (1-h)xE(\tilde{p}(\beta)) + p_f hx,$$

and

$$\sigma = (1-h)x\sigma_{\tilde{p}(\beta)},$$

respectively. The objective function becomes

$$U\Big((1-h)xE(\tilde{p}(\beta)) + p_f hx,\, (1-h)x\sigma_{\tilde{p}(\beta)})\Big).$$

By using risk aversion measure S and standard deviation σ the first order condition of the hedging decision problem becomes ($h(\beta) \neq 1$):

$$\left(p_f - E(\tilde{p}(\beta)) + \frac{\sigma S}{(1-h(\beta))}\right) U_\mu = 0,$$

which will be satisfied if and only if the term in brackets vanishes, since $U_\mu > 0$. With backwardation, i.e., $p_f < E(\tilde{p}(\beta))$, we obtain $h(\beta) < 1$ for the relevant range of β. The implicit function theorem then gives

$$\begin{aligned} \operatorname{sign}\left(\frac{dh(\beta)}{d\beta}\right) &= \operatorname{sign}\left(\frac{1}{1-h(\beta)}\left\{\frac{\partial\sigma}{\partial\beta}S + \sigma\frac{\partial S}{\partial\sigma}\frac{\partial\sigma}{\partial\beta}\right\}\right) \\ &= \operatorname{sign}\left(S + \sigma\frac{\partial S}{\partial\sigma}\right), \end{aligned}$$

since $1 - h(\beta) > 0$ and $\partial\sigma/\partial\beta > 0$. Applying the definition of elasticity of the risk aversion we obtain $\text{sign}[dh(\beta)/d\beta] = \text{sign}[1 - \varepsilon_{S,\sigma}]$. □

Note that the (μ, σ)-decision model is not in conflict with maximizing expected utility but has notably attractive properties. For example, it can be shown by the findings of Schneeweiß (1967), Sinn (1983), Meyer (1987), and Lajeri and Nielsen (2000) that the elasticity of risk aversion is always less than unity if Bernoulli-preferences display decreasing absolute risk aversion in the sense of Arrow and Pratt.

3. Concluding remarks

We have analyzed the optimum hedging policy of a firm when price risk changes. It is shown that risk aversion elasticity determines whether or not a (μ, σ)-risk averse firm (or, a risk averse expected utility maximizing firm) decreases or increases its optimum hedge ratio when market prices become more volatile. This is a remarkable direct characterization of the risk effect.

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