ECONOMICS BULLETIN

The Maschler−Perles Solution: 2 Simple Proofs for Superadditivity

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Abstract

The Maschler−−Perles Solution is the unique bargaining solution which is superadditive and satisfies the usual covariance properties. We provide two proofs for supperadditivity that do not rely on the standard traveling time.

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1 The MASCHLER–PERLES Solution

The MASCHLER–PERLES bargaining solution (MASCHLER–PERLES [2], [3], see also [5] for a textbook presentation) is a mapping defined on 2–dimensional bargaining problems respecting anonymity, Pareto efficiency, and affine transformations of utility. Moreover, this mapping is *superadditive* by which property it is uniquely characterized.

A ***bargaining problem*** is a pair $(\mathbf{0}, U)$ with a compact, convex, and comprehensive subset $\emptyset \neq U \subseteq \mathbb{R}^2_+$. $\mathbf{0}$ is the ***status quo point*** and U the ***feasible set***. Players may reach agreement on some feasible utility vector. Or else, they may fail to do so in which case they receive zero utility each. We omit reference to the status quo point (hence mention U only) as all concepts to be treated are covariant with "affine transformations of utility". A ***solution*** is a mapping $\boldsymbol{\varphi}$ that, based on some axiomatic justification, assignes to each bargaining problem U a Pareto efficient vector $\boldsymbol{\varphi}(U)$. Suppose two players (corporations, governments) are engaged in two "remote" problems U and U' simultaneously (one in Brussels and one in Washington). Originally, they considered these to be different affairs, thus there was a tendency to settle for $\boldsymbol{\varphi}(U)$ and $\boldsymbol{\varphi}(U')$ separately. Later on a *junctim* evolved and government officials considered giving in w.r. to one contract in favor of receiving concessions in the other one. The utilities available are now $\{x + x' | x \in U, x' \in U'\} \; =: \; U + U'$. The solution being superadditive, i.e., $\boldsymbol{\varphi}(U + U') \geq \boldsymbol{\varphi}(U) + \boldsymbol{\varphi}(U')$, it turnes out that both players profit from a *quid quo pro.*

Another interpretation is that players may face a lottery involving two bargaining problems. Superaddivity is then seen to consistently favor contracting *ex ante*, thereby increasing expected utility (see [2] or [5], *p.562*, for a detailed discussion).

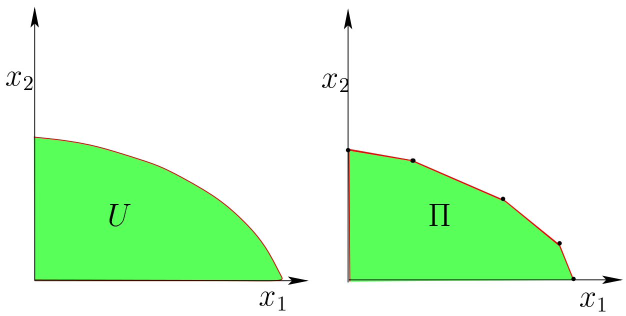

Figure 1.1: Bargaining problems – smooth and polyhedral

A bargaining problem is ***polyhedral*** if the Pareto surface consists of line segments only. The Maschler–Perles solution $\boldsymbol{\mu}$ is based on the observation that every polyhedral bargaining solution in $\mathbb{R}^2$ is an sum of "elementary" bargaining problems that are generated by a line segment (thus reflect constant transfer of utility). By continuity with respect to the Hausdorff metric the solution is transferred to bargaining problems with a smooth Pareto curve. More precisely, let $\boldsymbol{a} = (a_1, a_2) > 0 \in \mathbb{R}^n$. We introduce the unit vectors $\boldsymbol{e}^i$, the vectors $\boldsymbol{a}^i \; := \; a_i \boldsymbol{e}^i \;\; (i \in \boldsymbol{I})$, and associate with $\boldsymbol{a}$ the ***triangle***

$$\Pi^{\boldsymbol{a}} \; := \; \operatorname{convH}\left(\left\{\mathbf{0}, \boldsymbol{a}^1, \boldsymbol{a}^2, \right\}\right). \tag{1.1}$$

The Pareto curve of this triangle is the ***line segment*** $\Delta^{\boldsymbol{a}}$ which is given by

$$\Delta^{\boldsymbol{a}} \; := \; \operatorname{convH}\left(\left\{\boldsymbol{a}^1, \boldsymbol{a}^2\right\}\right). \tag{1.2}$$

A vector $\boldsymbol{a}$ is ***dyadic*** (with basis T) if there are integers t_1, t_2 such that $\boldsymbol{a} = \left(\frac{t_1}{2^T}, \frac{t_2}{2^T}\right)$ holds true.

Now, a bargaining problem is seen to be polyhedral if and only if the feasible set given by

$$\Pi = \sum_{k \in \boldsymbol{K}} \Pi^{a^{(k)}} \tag{1.3}$$

with a family of positive vectors

$$\left(\boldsymbol{a}^{(k)}\right)_{k \in \boldsymbol{K}}, \quad \boldsymbol{K} := \{1, \ldots, K\}$$

As it is sufficient to establish the solution on a dense subset with respect to the Hausdorff topology, we restrict ourselves to generating vectors $\boldsymbol{a}^{(k)}$ which are *dyadic* (with the same basis).

A triangle $\Pi^{\boldsymbol{a}}$ my be represented as ("homothetic") sum of two of its copies shrinked by suitable factor. In particular, we have

$$\Pi^{\boldsymbol{a}} = \frac{1}{2}\Pi^{\boldsymbol{a}} + \frac{1}{2}\Pi^{\boldsymbol{a}} = \Pi^{\frac{1}{2}\boldsymbol{a}} + \Pi^{\frac{1}{2}\boldsymbol{a}}.$$

By this operation the ***volume*** $V(\boldsymbol{a}) := a_1 a_2 = \frac{1}{2}$ area $(\Pi^{\boldsymbol{a}})$ is divided by 4, i.e.,

$$V(\frac{1}{2}\boldsymbol{a}) = \frac{1}{4}V(\boldsymbol{a}).$$

Therefore, we may assume that all triangles involved in a representation (1.3) have equal volume. The bargaining problems having this property again form a dense subset of the set of all bargaining problems. Similarly, whenever we deal with the sum of two bargaining problems, we may assume that the summands as well as the sum are dyadic with the same basis.

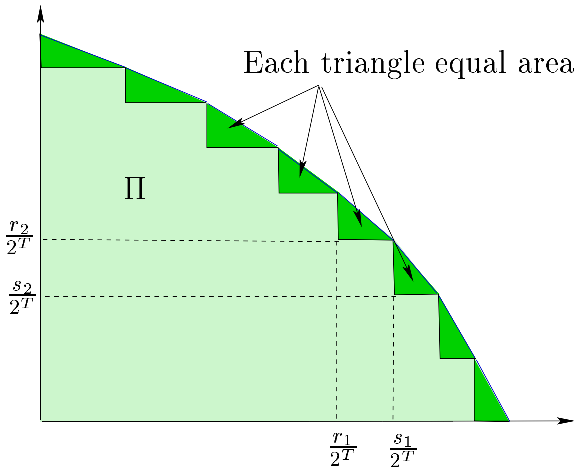

Figure 1.2: A standard dyadic bargaining problem

Definition 1.1. *We call a bargaining problem* ***standard dyadic*** *if the feasible set is a polyhedron represented as in* (1.3) *with dyadic vectors all generating equal volume.*

We assume the enumeration of the triangles to be such that the *tangents* (i.e., the quotients $\frac{a_2^{(k)}}{a_1^{(k)}}$) are decreasing with the index k. The Maschler–Perles solution for a standard dyadic bargaining problem is then defined inductively as follows: For $K = 1$ it is the midpoint of the line segment (the Pareto curve). For $K = 2$ (and assuming that the two triangles are not homothetic) it is the unique vertex of $\Pi = \Pi^{(1)} + \Pi^{(2)}$. For $K \geq 3$ it is defined by the *recursive formula*

$$\begin{aligned} \boldsymbol{\mu}(\Pi) &= \boldsymbol{\mu}\left(\sum_{k\in\boldsymbol{K}} \Pi^{a^{(k)}}\right) \\ &:= \boldsymbol{\mu}\left(\Pi^{(1)} + \Pi^{(K)}\right) + \boldsymbol{\mu}\left(\sum_{k\in\boldsymbol{K}-\{1,K\}} \Pi^{a^{(k)}}\right). \end{aligned} \tag{1.4}$$

This formula in fact implies uniquenes of the solution on standard dyadic bargaining problems. For, every superadditive solution $\boldsymbol{\mu}$ is necessarily additive whenever the solutions of the two summands admit of a joint normal.

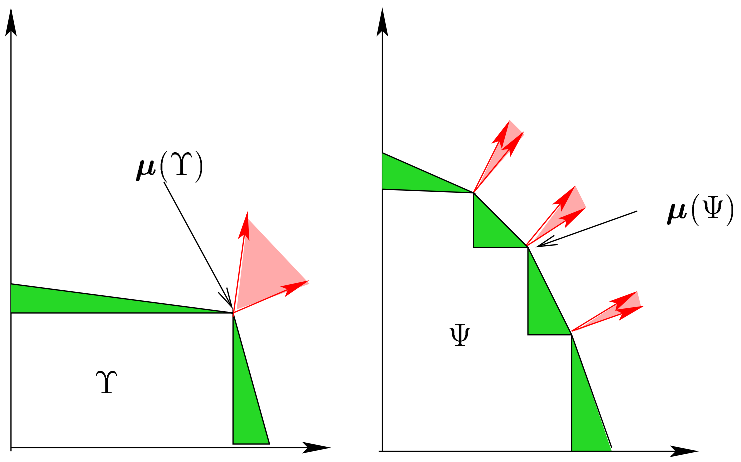

Figure 1.3: Additivity of the solution

To see this more clearly, consider Figure 1.3. Note that the sum of two Pareto efficient vectors is Pareto efficient if and only if both admit of a joint normal (equivalently: a joint tangency). In Figure 1.3, the corner point of Υ admits of a joint normal with *each* Pareto efficient point of Ψ (some normal cones are indicated). If the volumes of the two triangles involved in Υ are equal, then the solution yields this cornerpoint, denoted by $\boldsymbol{\mu}(\Upsilon)$, hence $\boldsymbol{\mu}(\Upsilon) + \boldsymbol{\mu}(\Psi)$ is Pareto efficient. As the solution is superadditive, we must necessarily have $\boldsymbol{\mu}(\Upsilon + \Psi) = \boldsymbol{\mu}(\Upsilon) + \boldsymbol{\mu}(\Psi)$.

In view of our enumeration, the first polyhedron $\Pi^{(1)} + \Pi^{(K)}$ plays the role of Υ, hence its solution admits of a joint normal with *every* Pareto efficient point of the second polyhedron.

2 Superadditivity

The recursive definition of the Maschler–Perles solution shows uniqueness at once. The fact that the solution is superadditive is proved by MASCHLER–PERLES [2], [3] using the concept of the "traveling time". To this end, they first extend the solution to smooth bargaining problems. The procedure amounts to having two points travel on the Pareto curve, simultaneously starting at each players "bliss point". The speed of the motion is arranged such that the product of velocities in axis directions is constant at each instant ("concessions" are made continuously in this way). When both points meet on the Pareto surface, the solution is reached. This concept is then carried back to polyhedral problems in order to show superadditivity. The reader may wish to consult MASCHLER–PERLES [2] or [5] (*CH. VIII, Theorem 4.21, p.588*) for more details.

We wish to provide two simple proofs that do not hinge on this concept. These versions can be used in classroom as they need no further preparation. Together, they might provide a clue for generalizing the concept to more than two dimensions. The first proof hinges on induction, thus it is close to the definition of the solution as discussed in SECTION 1. The second one is shorter and based on the ordering of slopes in a polyhedral probelem.

Theorem 2.1 (see [2], [3]). *Let* Π *be a dyadic polyhedron such that*

$$\Pi \;=\; \sum_{k\in \boldsymbol{K}} \Pi^{a^{(k)}} \tag{2.1}$$

holds true. Let

$$\Pi \;=\; \Upsilon + \Psi \tag{2.2}$$

where

$$\Upsilon \;=\; \sum_{k\in \boldsymbol{I}} \Pi^{a^{(k)}}\;, \quad \Psi \;=\; \sum_{k\in \boldsymbol{J}} \Pi^{a^{(k)}}. \tag{2.3}$$

Then

$$\boldsymbol{\mu}(\Pi) \geq \boldsymbol{\mu}(\Upsilon) + \boldsymbol{\mu}(\Psi). \tag{2.4}$$

Proof: By induction: if Π is the sum of two polyhedra (w.l.g. not homothetic) with equal volume, then $\boldsymbol{\mu}(\Pi)$ is the unique vertex on the Pareto surface of Π while $\boldsymbol{\mu}(\Upsilon) + \boldsymbol{\mu}(\Psi)$ is a non–Pareto efficient point on the line connecting $\mathbf{0}$ and $\boldsymbol{\mu}(\Pi)$.

In order to perform the induction step, assume that

$$\Pi \;=\; \Upsilon + \Psi, \quad \Upsilon \;=\; \sum_{k\in \boldsymbol{I}} \Pi^{a^{(k)}}\;, \quad \Psi \;=\; \sum_{k\in \boldsymbol{J}} \Pi^{a^{(k)}} \tag{2.5}$$

holds true.

1$^{\text{st}}$STEP : Assume that the indices 1 and K are jointly contained in one of the above sets, say $\{1, K\} \subseteq \boldsymbol{I}$. Then, as $\Pi^{(1)} + \Pi^{(K)}$ admits of joint normals at $\boldsymbol{\mu}(\Pi^{(1)} + \Pi^{(K)})$ with all other polyhedra involved, we have

$$
\begin{aligned}
\boldsymbol{\mu}(\Pi) &= \boldsymbol{\mu}\left(\Pi^{(1)} + \Pi^{(K)} + \sum_{k\in \boldsymbol{K}-\{1,K\}} \Pi^{a^{(k)}}\right) \\
&= \boldsymbol{\mu}\left(\Pi^{(1)} + \Pi^{(K)}\right) + \boldsymbol{\mu}\left(\sum_{k\in \boldsymbol{K}-\{1,K\}} \Pi^{a^{(k)}}\right) \\
&\qquad \text{(by Definition, see (1.4))} \\
&\geq \boldsymbol{\mu}\left(\Pi^{(1)} + \Pi^{(K)}\right) + \boldsymbol{\mu}\left(\sum_{k\in \boldsymbol{I}-\{1,K\}} \Pi^{a^{(k)}}\right) + \boldsymbol{\mu}\left(\sum_{k\in \boldsymbol{J}} \Pi^{a^{(k)}}\right) \\
&\qquad \text{(by induction hypothesis)} \\
&= \boldsymbol{\mu}\left(\sum_{k\in \boldsymbol{I}} \Pi^{a^{(k)}}\right) + \boldsymbol{\mu}\left(\sum_{k\in \boldsymbol{J}} \Pi^{a^{(k)}}\right) .
\end{aligned}
\tag{2.6}
$$

2nd STEP : Suppose now, that we have $1 \in \boldsymbol{I}$ and $K \in \boldsymbol{J}$. Let L denote the largest index in $\boldsymbol{I}$, i.e., the one wich induces the largest slope (in absolute value) of a line segment involved in Υ. Then we obtain

$$
\begin{aligned}
\boldsymbol{\mu}(\Pi) &= \boldsymbol{\mu}\left(\Pi^{(1)} + \Pi^{(L)} + \sum_{k\in \boldsymbol{I}-\{1,L\}} \Pi^{a^{(k)}} + \sum_{k\in \boldsymbol{J}} \Pi^{a^{(k)}}\right) \\
&\geq \boldsymbol{\mu}\left(\Pi^{(1)} + \Pi^{(L)} + \sum_{k\in \boldsymbol{J}} \Pi^{a^{(k)}}\right) + \boldsymbol{\mu}\left(\sum_{k\in \boldsymbol{I}-\{1,L\}} \Pi^{a^{(k)}}\right) \\
&\qquad \text{(by the } 1^{st} STEP \text{ as} \{1,K\} \subseteq \boldsymbol{J} + \{1,L\}) \\
&\geq \boldsymbol{\mu}\left(\Pi^{(1)} + \Pi^{(L)}\right) + \boldsymbol{\mu}\left(\sum_{k\in \boldsymbol{J}} \Pi^{a^{(k)}}\right) + \boldsymbol{\mu}\left(\sum_{k\in \boldsymbol{I}-\{1,L\}} \Pi^{a^{(k)}}\right) \\
&\qquad \text{(by induction hypothesis)} \\
&= \boldsymbol{\mu}\left(\sum_{k\in \boldsymbol{J}} \Pi^{a^{(k)}}\right) + \boldsymbol{\mu}\left(\Pi^{(1)} + \Pi^{(L)} \sum_{k\in \boldsymbol{I}-\{1,L\}} \Pi^{a^{(k)}}\right) \\
&\qquad \text{(by Definition applied to } \Upsilon\text{, see (1.4))} \\
&= \boldsymbol{\mu}\{\Upsilon\} + \boldsymbol{\mu}\{\Psi\},
\end{aligned}
\tag{2.7}
$$

q.e.d.

The second proof refers to the construction of the solution.

Proof: The enumeration is such that the tangent slope decreases with the index k. Since the *products* $a_1^{(k)} a_2^{(k)}$ are all equal, it follows that the enumeration satisfies

$$
\begin{array}{ll}
a_1^{(1)} \geq a_1^{(2)} \ \ldots & \geq a_1^{(K)} \ , \\
a_2^{(1)} \leq a_2^{(2)} \ \ldots & \leq a_2^{(K)}
\end{array}
\tag{2.8}
$$

W.l.o.g we may assume that K is even (otherwise split every polyhedron homothetically in two). Then we know that

$$\boldsymbol{\mu}(\Pi) = \left(\sum_{k=1}^{\frac{K}{2}} a_1^{(k)} \;,\; \sum_{k=\frac{K}{2}+1}^{K} a_2^{(k)} \right) , \tag{2.9}$$

that is, $\boldsymbol{\mu}(\Pi)$ *collects the* $\frac{K}{2}$ *largest vectors* with respect to each coordinate.

Now with respect to Υ we may as well assume that $|\boldsymbol{I}|$ is even. Thus, there is a decomposition $\boldsymbol{I} = \boldsymbol{I}_1 + \boldsymbol{I}_2$ with $|\boldsymbol{I}_1| = |\boldsymbol{I}_2|$ such that

$$\boldsymbol{\mu}(\Upsilon) = \left(\sum_{k\in\boldsymbol{I}_1} a_1^{(k)} \;,\; \sum_{k\in\boldsymbol{I}_2} a_2^{(k)} \right) . \tag{2.10}$$

The same holds true concerning Ψ with respect to a decomposition $\boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2$. Clearly, $|\boldsymbol{I}_1 + \boldsymbol{J}_1| = |\boldsymbol{I}_2 + \boldsymbol{J}_2| = \frac{K}{2}$ and hence

$$\boldsymbol{\mu}_1(\Upsilon) + \boldsymbol{\mu}_1(\Psi) = \sum_{k\in\boldsymbol{I}_1} a_1^{(k)} + \sum_{k\in\boldsymbol{J}_1} a_1^{(k)} = \sum_{k\in\boldsymbol{I}_1+\boldsymbol{J}_1} a_1^{(k)} \leq \sum_{k=1}^{\frac{K}{2}} a_1^{(k)} \tag{2.11}$$

as the last sum collects the largest $\frac{K}{2}$ coordinates (see (2.9)),

q.e.d.

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