E C O N O M I C S   B U L L E T I N

# Continuity of the payoff function revisited

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## *Abstract*

The payoff function is defined on the product of the spaces of mixed strategies that are the spaces of probability measures on compact Hausdorff spaces. The continuity of the payoff function is recently proved by Glycopantis and Muir. Here we give an alternative proof that is essentially based on existence of Milyutin maps.

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## 1. Introduction

For a compact Hausdorff space $X$, by $P(X)$ we denote the space of probability measures on $X$ endowed with the weak topology. Let $C(X)$ denote the space of continuous real-valued functions on $X$. For any fixed compact Hausdorff spaces, $K^1$ and $K^2$, and any $f \in C(K^1 \times K^2)$, let $\mathcal{E} = \mathcal{E}_f \colon P(K^1) \times P(K^2) \to \mathbb{R}$ be the function defined by the formula

$$\mathcal{E}(\mu, \nu) = \int_{K^1 \times K^2} f d(\mu \otimes \nu)$$

(here $\mu \otimes \nu$ denotes, as usual, the tensor product of $\mu \in P(K^1)$ and $\nu \in P(K^2)$).

The spaces $P(K^1)$ and $P(K^2)$ are interpreted as the sets of mixed strategies and the function $\mathcal{E}$ as the payoff function (the expected utility function).

The continuity of the function $\mathcal{E}_f$ is one of its essential properties. Glycopantis and Muir (2000) provided a proof of continuity, based on the Stone-Weierstrass theorem. In this note we give an alternative proof based on the existence of the so-called Milyutin maps (see the definition below). As Glycopantis and Muir (2000) do, for the sake of simplicity, we restrict ourselves with the case of two factors.

## 2. Preliminaries

Note that the construction of the space of probability measures is functorial onto the category of compact Hausdorff spaces: given a continuous map $f \colon X \to Y$ in this category, the map $Pf \colon P(X) \to P(Y)$ is defined by the condition

$$\int_Y \varphi dPf(\mu) = \int_X (\varphi \circ f) d\mu, \ \varphi \in C(Y).$$

It is well known that the map $Pf$ is continuous with respect to the weak topology.

The following property of the tensor products is also well known; we provide the proof for the sake of completeness.

**Lemma 2.1.** *Let $f_i \colon X^i \to Y^i$ be continuous maps of compact Hausdorff spaces, $\mu_i \in P(X^i)$, $i = 1, 2$. Then $P(f^1 \times f^2)(\mu_1 \otimes \mu_2) = Pf^1(\mu_1) \otimes Pf^2(\mu_2)$.*

*Proof.* If $A^i$ is a Borel subset of $Y^i$, $i = 1, 2$, then

$$\begin{aligned} &P(f^1 \times f^2)(\mu_1 \otimes \mu_2)(A^1 \times A^2) = (\mu_1 \otimes \mu_2)((f^1 \times f^2)^{-1}(A^1 \times A^2)) \\ =&(\mu_1 \otimes \mu_2)((f^1)^{-1}(A^1) \times (f^2)^{-1}(A^2))) = \mu_1((f^1)^{-1}(A^1))\mu_2((f^2)^{-1}(A^2)) \\ =&Pf^1(\mu_1)(A^1)Pf^2(\mu_2)(A^2) = (Pf^1(\mu_1) \otimes Pf^2(\mu_2))(A^1 \times A^2) \end{aligned}$$

and the required equality follows from the definition of the tensor product. □

The following lemma can be regarded as the property of naturality of the payoff function.

**Lemma 2.2.** *Let $f_i \colon X^i \to Y^i$ be continuous maps of compact Hausdorff spaces. Given $g \in C(X^1 \times X^2)$, $h \in C(Y^1 \times Y^2)$, denote by $\mathcal{E}_g \colon P(X^1 \times X^2) \to \mathbb{R}$, $\mathcal{E}'_h \colon P(Y^1 \times Y^2) \to \mathbb{R}$ the corresponding payoff functions. Then $\mathcal{E}_{h \circ (f_1 \times f_2)} = \mathcal{E}'_h \circ (Pf_1 \times Pf_2)$.*

*Proof.* Let $\mu_i \in P(X^i)$, $i = 1, 2$. Then

$$\begin{aligned}
\mathcal{E}_{h\circ(f_1\times f_2)}(\mu_1,\mu_2) &= \int_{X^1\times X^2} h\circ(f_1\times f_2)d(\mu_1\otimes\mu_2)\\
&= \int_{Y^1\times Y^2} hdP(f_1\times f_2)(\mu_1\otimes\mu_2)\\
\text{(by Lemma 2.1)}\ &= \int_{Y^1\times Y^2} hd(Pf_1(\mu_1)\otimes Pf_2(\mu_2))\\
&= \mathcal{E}'_h\circ(Pf_1\times Pf_2)(\mu_1,\mu_2).
\end{aligned}$$

□

There are different equivalent definitions of the Milyutin maps.

**Definition 2.1.** A continuous map $f\colon X \to Y$ of compact Hausdorff spaces is said to be *Milyutin* (see Pełczyński (1968)) if there exists a continuous map $\varphi\colon P(Y) \to P(X)$ such that $Pf \circ \varphi$ is the identity on $P(Y)$.

Recall that a space is *zero-dimensional* if it possesses a base consisting of sets that are simultaneously open and closed. It is known (see, e.g., Pełczyński (1968)) that for every compact Hausdorff space $Y$ there exists a Milyutin map $f\colon X \to Y$, where $X$ is a zero-dimensional compact Hausdorff space.

## 3. Result

**Theorem 3.1.** *The expected utility function $\mathcal{E}_f\colon P(K^1)\times P(K^2) \to \mathbb{R}$ is continuous.*

*Proof.* We consequently consider three cases.

1. $K^1$, $K^2$ are finite. The space $P(K^i)$ is naturally identified with $(|K^i|-1)$-dimensional symplex,

$$\sum_{j=1}^{|K^i|}\alpha_j\delta_{x^i_j} \mapsto (\alpha_1,\dots,\alpha_{|K^i|}),\ \ i=1,2.$$

The map $\mathcal{E}_f$ is in that case a polynomial map,

$$((\alpha_1,\dots,\alpha_{|K^1|}),(\beta_1,\dots,\beta_{|K^2|})) \mapsto \sum_{j=1}^{|K^1|}\sum_{l=1}^{|K^2|}\alpha_j\beta_l f(x^1_j,x^2_l),$$

and, therefore, continuous.

2. $K^1$, $K^2$ are zero-dimensional. Because of compactness and zero-dimensionality of $K^i$, for every $\varepsilon > 0$ there exist finite disjoint open covers (i.e., partitions), $\mathcal{V}^i_\varepsilon$, of $K^i$ such that the oscillation of $f$ on every set $V^1 \times V^2$, where $V^i \in \mathcal{V}^i_\varepsilon$, $i = 1, 2$, is less than $\varepsilon$.

Let $L^i_\varepsilon$ be the quotient space of $K^i$ with respect to the partition $\mathcal{V}^i_\varepsilon$ and $g^i_\varepsilon\colon K^i \to L^i_\varepsilon$ the quotient map. Define a function $f_\varepsilon\colon L^1_\varepsilon \times L^2_\varepsilon \to \mathbb{R}$ by putting $f_\varepsilon(x,y) = f(x',y')$, where $g^1_\varepsilon(x') = x$, $g^2_\varepsilon(y') = y$. Then, obviously, $\|f - f_\varepsilon\circ(g^1_\varepsilon\times g^2_\varepsilon)\| < \varepsilon$.

Denote by $\mathcal{E}'_{f_\varepsilon}\colon P(L^1_\varepsilon)\times P(L^2_\varepsilon) \to \mathbb{R}$ the expected utility map on $P(L^1_\varepsilon)\times P(L^2_\varepsilon)$ for the utility function $f_\varepsilon$. Then, by Lemma 2.2,

$$\mathcal{E}_{f_\varepsilon\circ(g^1_\varepsilon\times g^2_\varepsilon)}(\mu,\nu) = \mathcal{E}'_{f_\varepsilon}(Pg^1_\varepsilon(\mu),Pg^2_\varepsilon(\nu)),$$

and therefore, as a consequence of (established above) continuity of $\mathcal{E}'_{f_\varepsilon}$ and continuity of the maps $Pg^i_\varepsilon$, we obtain the continuity of $\mathcal{E}_{f_\varepsilon\circ(g^1_\varepsilon\times g^2_\varepsilon)}$ on $P(K^1)\times P(K^2)$. Since $\mathcal{E}_f$ is the uniform limit of the sequence $(\mathcal{E}_{f_{1/n}\circ(g^1_{1/n}\times g^2_{1/n})})$, it follows that $\mathcal{E}_f$ is continuous.

3. General case. Let $h^i\colon M^i \to K^i$ be Milyutin maps, where $M^i$ is zero-dimensional, $i = 1, 2$. Denote by $\varphi^i\colon P(K^i) \to P(M^i)$ a map such that $Ph^i \circ \varphi^i$ is the identity on

$P(K^i)$, $i=1,2$. Denote by $\mathcal{E}''_{f\circ(h^1\times h^2)}\colon P(M^1)\times P(M^2)\to\mathbb{R}$ the expected utility map corresponding to the utility function $f\circ(h^1\times h^2)$. Then, arguing similarly as above, we conclude that

$$\mathcal{E}''_{f\circ(h^1\times h^2)}(\varphi^1(\mu),\varphi^2(\nu))=\mathcal{E}_f(Ph^1\varphi^1(\mu),Ph^2\varphi^2(\nu))=\mathcal{E}_f(\mu,\nu)$$

and the continuity of the function $\mathcal{E}_f$ follows from the continuity of $\mathcal{E}''_{f\circ(h^1\times h^2)}$ established above in 2 and continuity of $Ph^i$. □

## 4. Concluding remarks

The continuity of the payoff functions is established by Glycopantis and Muir (2000) using one of the fundamental (and non-trivial) topological facts, the Stone-Weierstrass approximation theorem. Our proof is based on another non-trivial fact, the existence of Milyutin maps of a zero-dimensional compact Hausdorff space onto arbitrary compact Hausdorff space.

The main result can be generalized for the case of arbitrary set of players, finite or infinite. The proof in the case of infinite number of players requires technique of inverse systems (see, e.g., Shchepin (1981)).

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