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# Balanced-budget rules and aggregate instability: The role of endogenous capital utilization*

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**Abstract**

Schmitt-Grohé and Uribe (1997) demonstrate that a balanced-budget fiscal policy can induce aggregate instability unrelated to economic fundamentals. The empirical relevance of this result has been challenged by subsequent studies. In this paper we show, both analytically and numerically, that such extrinsic instability is an empirically robust plausibility associated with a balanced-budget rule once endogenous capital utilization is taken into consideration. This suggests that the design or operation of a balanced-budget fiscal policy must recognize that it may constitute a practical source of self-fulfilling prophecies and belief-driven fluctuations.

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*Keywords:* Balanced-budget rules; Income taxes; Public debt; Capital utilization; Consumption taxes; Sunspots; Self-fulfilling expectations; Indeterminacy

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# 1 Introduction

A recurrent debate in academic and policymaking circles is whether a government should operate under a balanced budget. On the one side, such fiscal discipline is considered a necessary tool to restrict deficit spending and limit government debt increment in order to ensure fiscal sustainability and long-run growth. On the other side, it is excoriated as a constraint on government's ability to deal with *fundamental shocks*, especially large adverse shocks such as wars and natural disasters, resulting in amplified, rather than dampened, short-run fluctuations, due to the cyclical nature of such policy (i.e., tax rate cut or government spending hike in boom but tax rate hike or government spending cut in bust). The benefit-cost comparison, along with certain operational considerations, has held a center stage in the debate surrounding a balanced-budget rule.[1] A clear understanding of the associated costs against its benefits is of critical importance in the consideration, design, or operation of a balanced-budget fiscal policy.[2]

The objective of this paper is to emphasize a cost associated with a balanced-budget rule which is not as publicized as the one highlighted above, but which may be more challenging to cope with, as it takes the form of *extrinsic instability unrelated to economic fundamentals*. The point that a balanced-budget rule which relies on adjusting income tax rates to finance government expenditures can be destabilizing, even in the absence of fundamental shocks, was first made by Schmitt-Grohé and Uribe (1997, SGU henceforth) using a canonical neoclassical model. In their model when agents anticipate lower (higher) labor income tax rates, the corresponding higher (lower) labor input would be coupled by higher (lower) capital input to produce higher (lower) total output, *provided that equilibrium output-labor elasticity is relatively high*, and then the income tax rates would indeed be called into being lowered (raised) along with the rise (decline) in total output, in order to maintain a balanced government budget. The countercyclical tax policy as such would render the agents' initial optimistic (pessimistic) expectations self-fulfilling and so is prone to equilibrium indeterminacy and belief-driven fluctuations. While this result has spurred a series of works,[3] its empirical relevance

[1] See Azzimonti (2013) for a recent survey of the debate and related literature.

[2] In practice, several European countries and all U.S. states other than Vermont have some forms of balanced budget provisions in their constitutions or basic laws, while there have also been repeated attempts to add a balanced-budget rule to the national United States Constitution.

[3] See, for example, Guo and Lansing (1998), Stockman (1998, 2010), Guillaume (1999), Pintus (2004), Guo and Harrison (2004, 2008), Gokan (2006, 2008, 2013), Giannitsarou (2007), Linnemann (2008), Dromel and Pintus (2008),

is challenged by subsequent studies.

A biggest and also most practical challenge goes back to the point of departure for advocating a balanced-budget rule, which has to do with concerns about excessive fiscal deficits and government debt sustainability. In actuality, a long period of sustained-deficit spending and\or a high debt to GDP ratio is a usual antecedent to a balanced-budget debate and oftentimes a trigger of the debate. However, if the government already has a large or even just a moderate debt outstanding, then a balanced-budget rule may no longer be prone to self-fulfilling expectations. This is demonstrated by Stockman (1998) and in fact also acknowledged by SGU (1997) themselves. Since in the absence of arbitrage interest rate is procyclical, as the rate of return on capital is, the pre-existing government borrowing entails a *procyclical debt payment*. When agents' optimism (pessimism) were to result in higher (lower) labor and capital inputs, debt payment would increase (decrease) along with the rise (decline) in total output and interest rate, thus government budget could be re-balanced without requiring the labor income tax rates to be lowered (raised). This would prevent the agents' initial optimistic (pessimistic) expectations from becoming self-fulfilling. The higher the debt to GDP ratio, the less likely it is for the economy to be destabilized by the balanced-budget fiscal policy rule. In particular, for the several countries examined by SGU (1997), once their government debt to GDP ratios are taken into account, the adoption of the balanced-budget rule in these countries would not induce any extrinsic instability originally concerned by SGU (1997) based on their model in which the debt to GDP ratio is set to zero.

It is a message of this paper that the issue raised by SGU (1997) can be empirically relevant, even with the pre-existing government debt taken into account, once another real-world feature is factored into consideration. We show that in this more realistic setting a balanced-budget rule is much more likely to induce extrinsic instability and it does so in an empirically plausible and robust manner. In fact, for those countries examined by SGU (1997), fluctuations in economic activities can emerge under a balanced-budget rule, at not only their current but much higher government debt to GDP ratios, even in the absence of fundamental shocks.

The real-world feature that is incorporated into our analysis is endogenous capital utilization. The concept of capital utilization as an optimal decision dates back to Keynes (1936), which has

Saïdi (2011), Nishimura et al. (2013), Nourry et al. (2013), Anagnostopoulos and Giannitsarou (2013), Ghilardi and Rossi (2014), Xue and Yip (2015), and Abad, et al. (2017).

been further developed by Taubman and Wilkinson (1970) and others. The essential ingredient is that increasing capital utilization increases the user cost of capital through an acceleration of capital depreciation. As a consequence, firms will not, in general, find it optimal to fully utilize the stock of capital, preferring to "hoard" some capital instead, so that they can use it more intensively when the returns to doing so are unusually large. Not only is this phenomenon much in line with the evidence documented in many empirical studies,[4] but respecting this realistic feature has proven important for deciphering a number of puzzles concerning growth and the business cycle.[5]

We show in the present paper, both analytically and numerically, that two important consequences of optimal capital utilization contribute to our result in this paper. First, it effectively increases equilibrium output-labor elasticity. Second, it makes interest rate and hence payment on government debt less responsive to aggregate output. As elaborated above, a relatively high equilibrium output-labor elasticity is a condition for a balanced-budget rule to help fulfill sunspot expectations, whereby a procyclical debt payment is a condition that helps make the balanced-budget rule immune to extrinsic instability. Endogenous capital utilization strengthens the former but weakens the latter. For the former, the optimal utilization of capital generates a redistribution of effective factor elasticities by increasing equilibrium output-labor elasticity and decreasing equilibrium output-capital elasticity. For the latter, as the marginal return to varying the utilization rate of capital is governed by the marginal product of capital, optimal decision prescribes that capital is used more intensively in boom when its marginal product is high but less in bust when its marginal product is low. The procyclical utilization rate results in a procyclical depreciation rate of capital. In consequence, interest rate, which in the absence of arbitrage must equal the rate of return on capital net of depreciation, becomes less responsive to aggregate output, and so does payment on government debt. These effects are quantitatively significant with empirically reasonable parametrization of our model.

The decisive role of endogenous capital utilization in rendering a balanced-budget rule susceptive of self-fulfilling prophecies is general and significant enough that it also offsets much of the effect of another practically relevant stabilizer, namely, consumption taxes.[6] As Giannitsarou (2007) shows,

[4]See Chatterjee (2005) for a survey.

[5]See, among others, Greenwood et al. (1988), Basu (1996), Burnside and Eichenbaum (1996), Burnside et al. (1996), Basu and Kimball (1997), Wen (1998), Dalgaard (2003), Chatterjee (2005), and Basu et al. (2006).

[6]Giannitsarou (2007) first demonstrates numerically the stabilization effect of consumption taxes in considering a

self-fulfilling beliefs may not arise from a balanced-budget rule if consumption taxes are included in the government's revenues; in particular, for the several countries examined by SGU (1997), once their consumption tax rates are taken into account, the adoption of the balanced-budget rule in these countries would not induce sunspot expectations originally concerned by SGU (1997) based on their model in which the consumption tax rate is set to zero. We show that, with endogenous capital utilization, extrinsic instability remains an empirically robust plausibility associated with a balanced-budget rule, even when we take account of consumption taxes, or of both consumption taxes and government debt, in the government's budget constraint. In fact, for those countries examined by SGU (1997), belief-driven fluctuations can emerge under a balanced-budget rule, not only at their current but much higher consumption tax rates or\and government debt to GDP ratios, even in the absence of fundamental shocks. This is to say that, the design or operation of a balanced-budget rule in these countries must recognize that it may constitute a practical source of self-fulfilling prophecies and sunspot-driven fluctuations.[7]

The rest of the paper is organized as follows. Section 2 presents the first set of the paper's main results for an economy with labor income taxes and public debt, while Section 3 presents the second set of the paper's main results for an economy with labor income and consumption taxes. Section 4 generalizes these analytical and numerical results to an environment with labor income and consumption taxes as well as public debt. Section 5 further generalizes the results, but only numerically, to a more realistic environment with not only labor income and consumption taxes and public debt, but also capital income taxes. Section 6 concludes. Most technical details, including the proofs of our six propositions, are relegated to the Appendix.

## 2 An economy with labor income taxes and public debt

This section derives analytically while also illustrating numerically the first set of the paper's main results. In order to do this our analytical framework integrates pre-existing government debt into

balanced-budget rule. Nourry et al. (2013) generalize the conclusion to all additively separable utility functions and they also prove their result analytically. These studies abstract from endogenous capital utilization.

[7]The literature has identified other potential sources of self-fulfilling expectations. There has also been an increased recognition of the practical relevance of belief-driven instability unrelated to economic fundamentals. For instance, sunspot expectations and belief coordination failures as a potential source of the recent financial crisis and ensuing recession is stressed by Farmer (2010) and also acknowledged by Lucas and Stokey (2011).

the baseline model of SGU (1997) with a balanced-budget labor income tax rule, augmented to include endogenous capital utilization.

Given initial capital stock $k_0$ and pre-existing stock of public debt $B$, the representative household chooses paths for consumption, $c_t$, hours worked, $l_t$, investment, $i_t$, capital stock, $k_t$, for $t > 0$, and capital utilization rate, $u_t$, to maximize the present discounted value of its lifetime utility,

$$\int_0^\infty (\log c_t - \eta l_t) \, e^{-\rho t} dt, \tag{1}$$

for marginal dis-utility from working $\eta > 0$,[8] and a subjective discount rate $\rho \in (0, 1)$, subject to,

$$c_t + i_t = w_t l_t + r_t (u_t k_t) + R_t B - T_t, \tag{2}$$

$$\dot{k}_t = i_t - \delta_t k_t, \tag{3}$$

$$T_t = \tau_t^l w_t l_t, \tag{4}$$

where $\tau_t^l$ denotes the labor income tax rate, $w_t$ the pretax wage rate, $r_t$ the rental rate for capital services, $R_t$ the rate of interest paid on public debt, and $\delta_t = \delta(u_t)$ the rate of capital depreciation, which is a function of the utilization rate of capital, with $\delta'(u_t) > 0$, $\delta''(u_t) \geq 0$. A parametrization that satisfies these properties takes the form, $\delta(u_t) = \tilde{\theta} u_t^\theta / \theta$, with $\theta > 1$ and $\tilde{\theta} > 0$ so that $\delta(u_t) \in [0, 1]$.[9] Denoting by $\lambda_t$ the costate variable, the resultant optimality conditions for these choices imply,

$$\lambda_t = c_t^{-1}, \tag{5}$$

$$(1 - \tau_t^l) w_t \lambda_t = \eta, \tag{6}$$

[8]The linearity of the period utility function in labor hours is a consequence of aggregation when labor is assumed to be indivisible and such a utility function is consistent with any labor supply elasticity at the individual level (e.g., Hansen 1985, Rogerson 1988). Such a formulation is commonly adopted in the RBC-based indeterminacy literature, including SGU (1997). See Benhabib and Farmer (1999) for an excellent survey.

[9]This approach in modeling the depreciation rate of capital as an increasing convex function of capacity utilization rate is similar to that in Taubman and Wilkinson (1970), Greenwood et al. (1988), and Jaimovich and Rebelo (2009) for a centralized system, but is more closely related to that in Greenwood and Huffman (1991), Finn (1995, 2000), and D'Erasmo et al. (2017) for a decentralized economy. It generalizes Keynes's notion of the user cost of capital – higher utilization causes faster depreciation, at an increasing rate, because of wear and tear on the capital stock. Note that here $\tilde{\theta} > 0$ is a scaling parameter that is adjusted to ensure that the steady state with variable capacity utilization is the same as the steady state with constant capacity utilization under a given balanced-budget rule.

$$\dot{\lambda}_t = \lambda_t \left[ \rho + \delta(u_t) - r_t u_t \right], \tag{7}$$

$$r_t = \delta'(u_t), \tag{8}$$

along with the following no-arbitrage condition,

$$R_t = r_t u_t - \delta(u_t), \tag{9}$$

which must hold in order to make the household indifferent between investing in the physical capital and holding the outstanding public debt.

Taking the wage and capital rental rates as given, the representative firm maximizes its profit,

$$y_t - w_t l_t - r_t(u_t k_t), \tag{10}$$

by hiring labor and capital services to produce the single good using a Cobb-Douglas technology,

$$y_t = (u_t k_t)^{\alpha} l_t^{1-\alpha}, \tag{11}$$

where $\alpha$ and $1-\alpha$ measure respectively output-capital service and output-labor elasticities. Perfect competition in factor and product markets implies the following optimality conditions,

$$w_t = (1-\alpha)\frac{y_t}{l_t}, \tag{12}$$

$$r_t = \alpha \frac{y_t}{u_t k_t}. \tag{13}$$

The government maintains a balanced budget each period,

$$G + R_t B = T_t = \tau_t^l w_t l_t, \tag{14}$$

by adjusting endogenously the labor income tax rate $\tau_t^l$ to generate enough revenue to finance the pre-set public spending $G$ and to make interest payment on the outstanding public debt $B$.[10]

Finally, the aggregate resource constraint for the economy is given by,

$$c_t + G + \dot{k}_t + \delta(u_t) k_t = y_t. \tag{15}$$

[10]We here follow SGU (1997) to assume that government expenditures are pre-set and constant and are for purchases of goods. The basic conclusions in this paper hold generally when we assume instead all distortionary tax revenues are rebated to the household in the form of lump-sum transfers.

In what follows, a variable with no time index denotes the steady-state value of that variable. As is shown in the Appendix, the local (in)stability property of this economy can be analyzed by examining the following system of two first-order linear differential equations,

$$\begin{pmatrix} \dot{k}_t \\ \dot{\lambda}_t \end{pmatrix} = \mathbf{J} \begin{pmatrix} k_t - k \\ \lambda_t - \lambda \end{pmatrix}, \tag{16}$$

where the elements of the two by two Jacobian matrix $\mathbf{J}$ are as specified in the Appendix. Since the system has one predetermined variable ($k_t$) and one jump variable ($\lambda_t$), it exhibits instability if and only if the two eigenvalues of $\mathbf{J}$ are both negative, whereas it is saddle-path stable if and only if the two eigenvalues are of opposite signs. The system has no equilibrium solutions that converge to the steady state if and only if the two eigenvalues are both positive.

Let $s \equiv [\rho + \delta(1-\alpha)]/(\rho+\delta)$, and denote by $s_b$ the steady-state public debt-GDP ratio $B/y$. We show in the Appendix that the following quadratic equation in $\tau^l$,

$$(1-\alpha)(\tau^l)^2 - (2s + \rho s_b)\tau^l + s = 0, \tag{17}$$

has two positive roots, one in $(0,1)$, denoted as $\overline{\tau}_b$, and the other in $(1,+\infty)$, and that the left side of (17) as a convex function of $\tau^l$, denoted as $\Omega_b(\tau^l)$, is strictly decreasing in $\tau^l$ for all $\tau^l \in [0,1]$. Further, let

$$\underline{\tau}_b^{VCU} \equiv \left[\frac{\rho}{\rho + \delta(1-\alpha)}\right] \underline{\tau}_b^{CCU}, \quad \text{where} \quad \underline{\tau}_b^{CCU} \equiv \alpha + (\rho+\delta)\, s_b. \tag{18}$$

The following proposition summarizes the first main result of this paper.

**Proposition 1.** *In the economy with variable capacity utilization where the government follows the balanced-budget rule (14), the equilibrium is indeterminate if and only if*

$$\underline{\tau}_b^{VCU} < \tau^l < \overline{\tau}_b. \tag{19}$$

Proposition 1 above establishes the indeterminacy region for the steady-state labor income tax rate for the economy with variable capacity utilization. It is worth noting that an economy with constant capacity utilization can be studied by recasting the above model appropriately. This is done by eliminating the first-order condition for $u_t$, (8), while setting $u_t = 1$ and $\delta_t = \delta$ throughout. The two economies have the same steady-state equilibrium under the same set of deep parameters.

**Proposition 2.** *In the economy with constant capacity utilization where the government follows the balanced-budget rule (14), the equilibrium is indeterminate if and only if*

$$\underline{\tau}_b^{CCU} < \tau^l < \overline{\tau}_b. \tag{20}$$

Proposition 2 above establishes the indeterminacy region for the steady-state labor income tax rate for the economy with constant capacity utilization. It should be obvious, in light of (18), that the indeterminacy region in (19) can only be greater than the indeterminacy region in (20). While the upper bound for the two regions is the same, the lower bound is lower for the region in (19) than in (20). Hence the economy can only be more likely to be destabilized by the balanced-budget rule (14) under variable capacity utilization than under constant capacity utilization.

To get a quantitative feel about how much larger the instability region in Proposition 1 is than that in Proposition 2, we now assign values to the deep parameters. Following SGU (1997), we set $\alpha = 0.3$, $\rho = 0.04$, and $\delta = 0.1$ for the model with constant capacity utilization. These are also the values chosen for the model with variable capacity utilization, where $\delta = 0.1$ corresponds to the steady-state capital depreciation rate which, given $\rho = 0.04$, is consistent with a 0.4 elasticity of marginal capital depreciation with respect to capacity utilization evaluated at the steady state, or, in terms of the parameterized capital depreciation function, $\theta = 1.4$, in the light of the steady-state equilibrium relation $\theta = 1 + \rho/\delta$.

Figure 1 plots the two instability regions established in Propositions 1 and 2 under the above parameter values, whereas equilibrium is determinate in the area outside the instability region. Just as displayed in (19) and (20), each instability region is characterized by a lower bound and an upper bound on the labor income tax rate (the horizontal axis) as a function of the public debt-GDP ratio (the vertical axis), and while the upper bound is common the lower bound is smaller under variable capacity utilization than under constant capacity utilization, conforming to the analytical result. As the figure shows, the reduction in the lower bound due to optimal use of capacity is significant, and it takes the form of both a shift and a rotation (to the left and counterclockwise in the figure). The shift renders the balanced-budget rule (14) destabilizing at much lower labor income tax rates, and the rotation greatly weakens the stabilization role played by the existing government debt. As a result, the instability region is nearly doubled by endogenizing the capital utilization rate.

This doubling in the instability region by endogenous capital utilization has important empirical implications. To put this into perspective, Figure 1 also displays the pairs of labor income tax rate and public debt-GDP ratio for the five countries studied by SGU (1997), including Canada (CA), Germany (GE), Japan (JP), the United Kingdom (UK), and the United States (US). These tax rates and debt-GDP ratios are summarized in Table 1. As can be seen from the figure, under constant capacity utilization, only Germany would fall into the instability region whereas all the other four countries would fall into the stability region. In contrast, under variable capacity utilization, all the five countries would fall into the instability region. This is true whether we apply the 1988 tax rates estimated by Mendoza, Razin, and Tesar (1994) and used by SGU (1997), or the 1996 tax rates updated by these authors and available online at their website. Hence, if endogenous capital utilization were ignored one would conclude that the adoption of the balanced-budget rule (14) would only destabilize the economy of Germany but not of the other four countries, whereas in actuality, when endogenous capital utilization is factored into consideration the adoption of such balanced-budget rule would destabilize all of the five economies, and this would happen at not only their current but much higher government debt to GDP ratios.

**Some intuition**

We now provide some intuition behind Propositions 1 and 2. We can prove analytically that the upper bound in (19) and (20) coincides with a labor income tax rate corresponding to the peak of a Laffer curve that represents the tax revenue as a function of the tax rate.[11] This result generalizes the insight about the connection between the shape and characteristics of a Laffer curve and indeterminacy under the balanced-budget rule and dependence of labor supply on tax rate, presented in SGU (1997) and Anagnostopoulos and Giannitsarou (2013), to an environment with public debt and endogenous or exogenous capital utilization. This explains why for indeterminacy to occur there must be a nonempty set of tax rates on the upward sloping side of a Laffer curve as characterized by the open interval in (19) or (20). However, since the two intervals in (19) and (20) have the same upper bound, the key to understanding why endogenous capital utilization makes the balanced-budget rule (14) more prone to indeterminacy is to understand why it lowers the lower bound in (19) below that in (20), in particular, why it generates both a shift and a rotation of the lower bound.

*The shift in the lower bound*

The shift is measured in Figure 1 by the distance between the intercepts of the two lower bounds with the horizontal axis, or, in light of (18), from $\alpha$ to $\alpha\rho/[\rho+\delta(1-\alpha)]$. As explained in the introduction, a high output-labor elasticity, or, equivalently, a low output-capital elasticity, relative

[11]The proof and demonstration of this result are not presented here due to the space constraint but are available upon request from the authors.

to the labor income tax rate, is a condition for the balanced-budget rule to help fulfill sunspot expectations. This is why $\alpha$, corresponding to the effective output-capital elasticity under constant capacity utilization, shows up as the first component of the lower bound $\underline{\tau}_b^{CCU}$. To understand why the first component of the lower bound $\underline{\tau}_b^{VCU}$ is a reduction from $\alpha$, to $\alpha\rho/[\rho+\delta(1-\alpha)]$, we note that optimal utilization of capital generates a redistribution of effective factor elasticities: It increases effective output-labor elasticity and decreases effective output-capital elasticity. To see this, we combine (8) and (13) to solve for the optimal capital utilization rate,

$$u_t = \left(\alpha \frac{y_t}{k_t}\right)^{\frac{\delta}{\rho+\delta}} = \alpha^{\frac{\delta}{\rho+\delta(1-\alpha)}} \left(\frac{l_t}{k_t}\right)^{\frac{\delta(1-\alpha)}{\rho+\delta(1-\alpha)}}, \tag{21}$$

which is an increasing function of labor but a decreasing function of capital. This implies that the optimal operation of capacity effectively decreases equilibrium output-capital elasticity and increases equilibrium output-labor elasticity. This can be seen more clearly by substituting (21) into (11) to derive the following reduced-form aggregate production function,

$$y_t = \alpha^{\frac{\delta\alpha}{\rho+\delta(1-\alpha)}} k_t^{\left[\frac{\rho}{\rho+\delta(1-\alpha)}\right]\alpha} l_t^{\left[\frac{\rho+\delta}{\rho+\delta(1-\alpha)}\right](1-\alpha)}. \tag{22}$$

Under the parameter values specified above, optimal capacity utilization lowers effective output-capital elasticity from 0.3 to near 0.1 and raises effective output-labor elasticity from 0.7 to around 0.9. This explains why the shift in the lower bound in Figure 1, that is, the distance between the intercepts of the two lower bounds with the horizontal axis, given by $\delta\alpha(1-\alpha)/[\rho+\delta(1-\alpha)]$, is almost as large as 0.2.

*The rotation of the lower bound*

The rotation is measured in Figure 1 by the difference between the slopes of the two lower bounds on labor income tax rate viewed as a function of public debt-GDP ratio. In light of (18), the slope is $(\rho+\delta)$ under constant capacity utilization, but $(\rho+\delta)\rho/[\rho+\delta(1-\alpha)]$ under variable capacity utilization. In both cases, the slope is positive so the lower bound on the tax rate rises with the debt-GDP ratio, since, as explained in the introduction, the pre-existing government borrowing entails a procyclical debt payment, which is a condition that helps make the balanced-budget rule immune to extrinsic instability. But the slope is smaller in the latter case than in the former, so the lower bound rises at a lower rate under variable capacity utilization than under constant capacity utilization. This is because endogenous capacity utilization, by generating a procyclical utilization rate and thus a procyclical depreciation rate of capital, makes interest rate and hence payment on government debt less responsive to aggregate output. This can be seen by using (8), (9) and (13), and the appropriate version of the latter two for the case with constant capacity utilization, to derive the responsiveness of interest rate to changes in aggregate output in the two cases,

$$\frac{\partial R_t}{\partial y_t} = \begin{cases} \frac{\alpha}{k_t} & \text{with constant capacity utilization,} \\ \left(\frac{\rho}{\rho+\delta}\right)\frac{\alpha}{k_t} & \text{with variable capacity utilization,} \end{cases} \tag{23}$$

where in deriving the second line in (23), we have used the following solution for capital depreciation rate that is consistent with the optimal capacity utilization rate characterized by (21),

$$\delta_t = \frac{\delta\alpha}{\rho+\delta}\left(\frac{y_t}{k_t}\right). \tag{24}$$

As is clear from (21) and (24), with endogenous capacity utilization, $u_t$ and $\delta_t$ vary directly with $y_t$ (where we recall that $k_t$ is pre-determined), since optimal decision prescribes that capital is used more intensively in boom when its marginal product is high but less in bust when its marginal product is low. Because interest rate is equal to the rate of return on capital net of depreciation, as governed by (9), the procyclical capital depreciation rate, as prescribed by (24), dampens the responsiveness of interest rate to aggregate output, by a factor of $\rho/(\rho+\delta)$, as shown by (23). This factor is 0.28 under the parameter values specified above. This translates into a factor of 0.36 when comparing the slopes of the two lower bounds on the labor income tax rate viewed as a function of the public debt-GDP ratio. In other words, endogenous capital utilization eliminates 64% of the effect of the public debt on the lower bound for the tax rate.

We can also relate our result more generally to a condition for indeterminacy first discovered by Benhabib and Farmer (1994). These authors show that, for indeterminacy to occur, the equilibrium labor demand schedule must be upward slopped and steeper than the labor supply schedule. In the present paper with indivisibility in labor, the labor supply schedule is flat, so indeterminacy simply requires that the equilibrium labor demand schedule be upward slopping. With some algebra, we can obtain the the slope of the equilibrium labor demand schedule as follows,

$$\text{Slope of labor demand schedule} = \begin{cases} \frac{\tau^l - \underline{\tau}_b^{CCU}}{1-\tau^l} & \text{with constant capacity utilization,} \\ \frac{\tau^l - \underline{\tau}_b^{VCU}}{1-\tau^l} & \text{with variable capacity utilization.} \end{cases} \tag{25}$$

In light of (25), the lower bounds in (19) and (20) on $\tau^l$ are equivalent to requiring the slope of labor demand schedule to be positive, under variable and constant capacity utilization, respectively.

Since $\underline{\tau}_b^{VCU} < \underline{\tau}_b^{CCU}$, the labor demand schedule is steeper under variable capacity utilization than under constant capacity utilization.

## 3 An economy with labor income and consumption taxes

The decisive role of endogenous capacity utilization in rendering a balanced-budget rule susceptive of extrinsic instability is general and significant enough that it also offsets much of the effect of another practically relevant stabilizer, that is, consumption taxes. In this section, we modify the model presented in Section 2 by introducing consumption taxes while setting public debt to zero in order to sharpen the result. The household budget constraint (2) and the government revenue (4) are then amended respectively as,

$$c_t + i_t = w_t l_t + r_t (u_t k_t) - T_t, \tag{26}$$

$$T_t = \tau_t^l w_t l_t + \tau_t^c c_t, \tag{27}$$

where $\tau_t^c$ denotes the consumption tax rate, the first order condition (5) changes to,

$$(1 + \tau_t^c)\lambda_t = c_t^{-1}, \tag{28}$$

while (9) is eliminated, and the government budget constraint (14) is amended as,

$$G = T_t = \tau_t^l w_t l_t + \tau_t^c c_t. \tag{29}$$

The other features remain the same as in the baseline model presented in Section 2.

We show in the Appendix that there exists a unique root $\overline{\tau}_c \in (0, 1)$ that solves the following equation in $\tau^l$,

$$\frac{(1-\alpha)(\tau^l)^3 + [(1-\alpha)\tau^c - 2s](\tau^l)^2 + s\left[(\tau^c)^2 - \frac{s(2+\tau^c)\tau^c}{1-\alpha} + 1\right]\tau^l + \frac{s^2\tau^c}{1-\alpha}}{(1+\tau^c)^2\tau^l} = 0, \tag{30}$$

and that the left side of (30) as a function of $\tau^l$, denoted as $\Omega_c(\tau^l)$, is strictly decreasing in $\tau^l$ for $\tau^l \in (0, 1]$.

In the Appendix we also show that there are a negative root and a positive root lying strictly between 0 and 1 for each of the following two quadratic equations,

$$\left[\frac{\rho + (1-\alpha)\delta}{\alpha\rho} - \frac{\tau^c}{(1+\tau^c)^2}\right](\tau^l)^2 + \left[\left(1 + \frac{s}{1-\alpha}\right)\frac{\tau^c}{(1+\tau^c)^2} - 1\right]\tau^l - \frac{s\tau^c}{(1-\alpha)(1+\tau^c)^2} = 0, \tag{31}$$

$$\left[\frac{1}{\alpha} - \frac{\tau^c}{(1+\tau^c)^2}\right](\tau^l)^2 + \left[\left(1 + \frac{s}{1-\alpha}\right)\frac{\tau^c}{(1+\tau^c)^2} - 1\right]\tau^l - \frac{s\tau^c}{(1-\alpha)(1+\tau^c)^2} = 0, \tag{32}$$

and that the left sides of (31) and (32) as convex functions of $\tau^l$, denoted as $\Pi_c^{VCU}(\tau^l)$ and $\Pi_c^{CCU}(\tau^l)$, are strictly increasing in $\tau^l$, for $\tau^l \in [\underline{\tau}_c^{VCU}, 1]$ and $\tau^l \in [\underline{\tau}_c^{CCU}, 1]$, where $\underline{\tau}_c^{VCU}$ and $\underline{\tau}_c^{CCU}$ denote the positive roots that solve (31) and (32), respectively. We show in the Appendix that,

$$\underline{\tau}_c^{VCU} < \underline{\tau}_c^{CCU}. \tag{33}$$

The following two propositions summarize the second set of this paper's main results.

**Proposition 3.** *In the economy with variable capacity utilization where the government follows the balanced-budget rule (29), the equilibrium is indeterminate if and only if*

$$\underline{\tau}_c^{VCU} < \tau^l < \overline{\tau}_c. \tag{34}$$

**Proposition 4.** *In the economy with constant capacity utilization where the government follows the balanced-budget rule (29), the equilibrium is indeterminate if and only if*

$$\underline{\tau}_c^{CCU} < \tau^l < \overline{\tau}_c. \tag{35}$$

According to (33), the lower bound for the indeterminacy region in (34) is lower than the lower bound for the indeterminacy region in (35), though the upper bound for the two regions is the same. Thus the indeterminacy region in (34) can only be greater than the indeterminacy region in (35). This is to say that the economy can only be more likely to be destabilized by the balanced-budget rule (29) under variable capacity utilization than under constant capacity utilization.

To see quantitatively how much greater the instability region in Proposition 3 is than that in Proposition 4, Figure 2 plots the two instability regions under the calibrated parameter values, whereas equilibrium is determinate in the area outside the instability region. Just as (34) and (35) demonstrate, each instability region is characterized by a lower bound and an upper bound on the labor income tax rate (the horizontal axis) as a function of the consumption tax rate (the vertical axis), and while the upper bound is common the lower bound is smaller under variable capacity utilization than under constant capacity utilization, conforming to the analytical result. As the figure illustrates, the reduction in the lower bound due to optimal capacity utilization significantly enlarges the instability region.

This enlargement of the instability region by endogenous capital utilization has once again important empirical implications. To put this into perspective, Figure 2 also displays the pairs of labor income and consumption tax rates for the five countries studied in Section 2. As can be seen from the figure, under constant capacity utilization, only Germany would fall into the instability region whereas Canada, Japan, the United Kingdom, and the United States would all fall into the stability region. In contrast, under variable capacity utilization, all the five countries would fall into the instability region. This is true whether we apply the 1988 or the 1996 tax rates. Therefore, if endogenous capital utilization were ignored one would conclude that the adoption of the balanced-budget rule (29) would only destabilize the economy of Germany but not of the other four countries, whereas in actuality, when endogenous capital utilization is factored into consideration the adoption of such balanced-budget rule would destabilize all of the five economies, and this would happen at not only their current but much higher consumption tax rates.

# 4 With labor income and consumption taxes and public debt

In this section, we analyze an economy that encompasses those in Sections 2 and 3 as special cases. This is done by incorporating consumption taxes into the government revenue in Section 2, so (4) becomes (27), (5) changes to (28), and the government budget constraint (14) is amended as,

$$G + R_t B = T_t = \tau_t^l w_t l_t + \tau_t^c c_t. \tag{36}$$

The other features remain the same as in the baseline model presented in Section 2.

We show in the Appendix that there exists a unique root $\overline{\tau}_{b,c} \in (0,1)$ that solves the following equation in $\tau^l$,

$$\begin{aligned} &\frac{(1-\alpha)(\tau^l)^3 + \left[(1-\alpha)\tau^c - 2s - \rho s_b\right](\tau^l)^2}{(1+\tau^c)^2\tau^l} \\ &+ \frac{\left\{s\left[1+(\tau^c)^2\right] - \frac{s(s+\rho s_b)(2+\tau^c)\tau^c}{1-\alpha} - \rho s_b \tau^c\right\}\tau^l + \frac{s(s+\rho s_b)\tau^c}{1-\alpha}}{(1+\tau^c)^2\tau^l} = 0, \end{aligned} \tag{37}$$

and that the left side of (37) as a function of $\tau^l$, denoted as $\Omega_{b,c}(\tau^l)$, is strictly decreasing in $\tau^l$ for $\tau^l \in (0,1]$.

In the Appendix we also show that there are a negative root and a positive root for each of the following two quadratic equations,

$$\begin{aligned}
&\left[\frac{\rho+(1-\alpha)\delta}{\alpha\rho}-\frac{\tau^c}{(1+\tau^c)^2}\right](\tau^l)^2 \qquad (38)\\
+\ &\left[\left(1+\frac{s}{1-\alpha}\right)\frac{\tau^c}{(1+\tau^c)^2}+\left(\frac{\rho\tau^c}{(1-\alpha)(1+\tau^c)^2}-\frac{\rho+\delta}{\alpha}\right)s_b-1\right]\tau^l-\frac{(s+\rho s_b)\tau^c}{(1-\alpha)(1+\tau^c)^2}=0,
\end{aligned}$$

$$\begin{aligned}
&\left[\frac{1}{\alpha}-\frac{\tau^c}{(1+\tau^c)^2}\right](\tau^l)^2 \qquad (39)\\
+\ &\left[\left(1+\frac{s}{1-\alpha}\right)\frac{\tau^c}{(1+\tau^c)^2}+\left(\frac{\rho\tau^c}{(1-\alpha)(1+\tau^c)^2}-\frac{\rho+\delta}{\alpha}\right)s_b-1\right]\tau^l-\frac{(s+\rho s_b)\tau^c}{(1-\alpha)(1+\tau^c)^2}=0,
\end{aligned}$$

and that the left sides of (38) and (39) as convex functions of $\tau^l$, denoted as $\Pi_{b,c}^{VCU}(\tau^l)$ and $\Pi_{b,c}^{CCU}(\tau^l)$, are strictly increasing in $\tau^l$, for $\tau^l \in [\underline{\tau}_{b,c}^{VCU}, +\infty)$ and $\tau^l \in [\underline{\tau}_{b,c}^{CCU}, +\infty)$, where $\underline{\tau}_{b,c}^{VCU}$ and $\underline{\tau}_{b,c}^{CCU}$ denote the positive roots that solve (38) and (39), respectively. We show in the Appendix that,

$$\underline{\tau}_{b,c}^{VCU} < \underline{\tau}_{b,c}^{CCU}. \qquad (40)$$

The following two propositions generalize the first two sets of the paper's main results.

**Proposition 5.** *In the economy with variable capacity utilization where the government follows the balanced-budget rule (36), the equilibrium is indeterminate if and only if*

$$\underline{\tau}_{b,c}^{VCU} < \tau^l < \overline{\tau}_{b,c}. \qquad (41)$$

**Proposition 6.** *In the economy with constant capacity utilization where the government follows the balanced-budget rule (36), the equilibrium is indeterminate if and only if*

$$\underline{\tau}_{b,c}^{CCU} < \tau^l < \overline{\tau}_{b,c}. \qquad (42)$$

While the upper bound for the two indeterminacy regions in (41) and (42) is the same, inequality (40) shows that the lower bound in (41) is lower than the lower bound in (42). This is to say that the indeterminacy region in (41) can only be greater than the indeterminacy region in (42). Therefore, the economy can only be more likely to be destabilized by the balanced-budget rule (36) under variable capacity utilization than under constant capacity utilization. This generalizes the results that have been established in Sections 2 and 3 to an environment featuring both public debt and consumptions taxes, along with labor income taxes.

Figure 3 puts this into a quantitative perspective by plotting the two instability regions in (41) and (42) under the calibrated parameter values, whereas equilibrium is determinate in the area outside the instability region. Conforming to the analytical results in Propositions 5 and 6, each instability region is characterized by a lower bound and an upper bound on the labor income tax rate as a function of the public debt-GDP ratio and the consumption tax rate, and while the upper bound is common the lower bound is smaller under variable capacity utilization than under constant capacity utilization. As the figure shows, the reduction in the lower bound due to optimal capacity utilization dramatically expands the instability region, and this expansion has important empirical implications. As can be seen from the figure (which also displays the triples of labor income and consumption tax rates and public debt-GDP ratio for the five countries studied above), under constant capacity utilization, only Germany would fall into the instability region whereas Canada, Japan, the UK, and the US would all fall into the stability region. Under variable capacity utilization, in contrast, all of the five countries would fall into the instability region. This is true for both the 1988 and the 1996 tax rates. Thus, if endogenous capital utilization were ignored one would conclude that the adoption of the balanced-budget rule (36) would only destabilize the economy of Germany but not of the other four countries, whereas in actuality, when endogenous capital utilization is taken into account the adoption of such balanced-budget rule would destabilize all of the five economies, and this would happen at not only their current but much higher consumption tax rates or\and public debt-GDP ratios.

# 5 Incorporating capital income taxes

In this section we introduce capital income taxes into the economy in Section 4 so as to make the model's tax structure closer to the one observed in the real world. With capital income taxes, transparent analytical results are hard to come by, but the numerical exercises below reinforce our conclusions based on the analytical results obtained in the previous sections that abstract from capital income taxes. To be specific about this general setting, the government revenue and budget constraint are now given by,[12]

$$T_t = \tau_t^l w_t l_t + \tau_t^k r_t u_t k_t + \tau_t^k R_t B + \tau_t^c c_t, \tag{43}$$

$$G + \left(1 - \tau_t^k\right) R_t B = \tau_t^l w_t l_t + \tau_t^k r_t u_t k_t + \tau_t^c c_t, \tag{44}$$

where $\tau_t^k$ denotes the capital income tax rate, and the no-arbitrage condition (9) is modified as,

$$R_t = r_t u_t - \frac{\delta(u_t)}{1 - \tau_t^k}. \tag{45}$$

The other features remain the same as in the model presented in Section 4.

In the general setting with labor and capital income as well as consumption taxes, along with public debt, the contrast between the implications for aggregate instability across constant and variable capacity utilization presented above continues to hold for the balanced-budget rule (44). Our numerical exercises reaffirm the analytical results obtained in Sections 2, 3, and 4 that abstract from capital income taxes, and the results from these numerical exercises are reported in Figure 4, which plots the instability region under constant capacity utilization against the instability region under variable capacity utilization for the calibrated parameter values in this general setting.

There are five rows in Figure 4, each corresponding to one of the five countries studied above. In each panel of a given row, each of the two instability regions is characterized by a lower bound and an upper bound, whereas equilibrium is determinate in the area outside the instability region. The bounds are on the labor income tax rate (the horizontal axis) as a function of the capital income tax rate (the vertical axis), where the bounds themselves are functions of the consumption tax rate and public debt-GDP ratio of the country represented by the row, in addition to the model's deep parameters. Whereas the upper bound is common, the lower bound is significantly smaller

[12]When capital income taxes are considered, existing studies abstracting from endogenous capacity utilization reveal that the balanced-budget rule is less susceptive of sunspot beliefs if no depreciation in capital is tax-deductible than if deduction is allowed (e.g., SGU 1997, Giannitsarou 2007). Given that our goal in this paper is to show how endogenous capacity utilization renders the balanced-budget rule destabilizing, we start from this point that makes indeterminacy harder to occur. All of our results hold also for the case with depreciation allowance, where (43), (44), and (45) become $T_t = \tau_t^l w_t l_t + \tau_t^k (r_t u_t - \mu\delta_t) k_t + \tau_t^k R_t B + \tau_t^c c_t$, $G + \left(1 - \tau_t^k\right) R_t B = \tau_t^l w_t l_t + \tau_t^k (r_t u_t - \mu\delta_t) k_t + \tau_t^c c_t$, and $R_t = r_t u_t - \delta(u_t)(1 - \mu\tau_t^k)/(1 - \tau_t^k)$, where $\mu \in [0, 1]$ represents the degree of depreciation allowance. Note that, for the case with full depreciation allowance (i.e., $\mu = 1$), the no-arbitrage condition coincides with (9), while for the case with no depreciation allowance (i.e., $\mu = 0$), it reduces to (45). Details of these additional results are not presented here in order to conserve space but available upon request from the authors.

under variable capacity utilization than under constant capacity utilization. This generalizes the analytical results in Sections 2, 3, and 4 to a more realistic environment with not only labor income and consumption taxes and public debt, but capital income taxes. Once again, as is clear from the figure, the reduction in the lower bound due to optimal capacity utilization drastically enlarges the instability region. As a result, an economy is much more likely to be destabilized by the balanced-budget rule (44) under variable capacity utilization than under constant capacity utilization.

The left and right panels in each row of Figure 4 also display the pairs of the labor and capital income tax rates effective in 1988 and 1996, respectively, for the country that the row represents, while recalling that the country's consumption tax rate and debt-GDP ratio are already embedded in the lower and upper bounds in the corresponding panel of the figure. As is clear from the figure, under constant capacity utilization, only Germany would fall into the instability region whereas Canada, Japan, the UK, and the US would all fall into the stability region. In contrast, under variable capacity utilization, all of the five countries would fall into the instability region. This is true for both the 1988 or the 1996 tax rates. This is to say that, if endogenous capital utilization were ignored, one would conclude that the adoption of the balanced-budget rule (44) would only destabilize the economy of Germany but not of the other four countries, whereas in actuality, when endogenous capital utilization is factored into consideration the adoption of such balanced-budget rule would destabilize all of the five economies. In fact, this would happen at not only their current but much higher consumption tax rates or\and public debt-GDP ratios.

Table 2 summarizes the (in)stability implications of the various versions of a balanced-budget fiscal policy that have been studied in Sections 2 through 5 for the five countries under constant and variable capacity utilization respectively. The table serves to remind the reader of the consistent message throughout this paper concerning the decisive role of endogenous capital utilization in rendering a balanced-budget rule destabilizing. Since capital utilization as an optimal decision is a widespread empirical phenomenon, our results suggest that the consideration of a balanced-budget rule in these countries must recognize that it may constitute a practical source of self-fulfilling prophecies and sunspot-driven fluctuations.

We have adopted a canonical continuous-time setting to facilitate the analytical derivations, but our results and basic conclusions hold quite generally and in a discrete-time setup as well.[13]

[13]These additional results are not presented in the present paper in order to conserve space, but they are available

The broad implications of our results in this paper issue a caution for the consideration, design, or operation of a balanced-budget fiscal policy.

## 6 Concluding remarks

We have shown that expectation-driven fluctuations unrelated to economic fundamentals can be a practical issue associated with a balanced-budget rule. Such belief-induced, extrinsic instability is an empirically robust plausibility once endogenous capital utilization is taken into consideration. This poses a unique challenge to governments in their consideration or operation of balanced-budget fiscal policies. Previous studies suggest that monetary policies can be an especially effective tool to preempt sunspot expectations and stabilize belief-driven fluctuations. This suggests that one promising approach in meeting the above challenge may require coherent designs of and proper interactions between fiscal and monetary policies. We leave this topic for future research.

upon request from the authors.

# A Appendix

In this appendix we sketch our proofs of Propositions 1-6 and Inequalities (33) and (40). The local instability analysis in each case boils down to analyzing the two by two dynamic system (16), where the Jacobian matrix **J** differers across the six cases. Since the system has one predetermined variable and one jump variable, it exhibits instability if and only if the two eigenvalues of **J** are both negative, that is, if and only if the trace of **J** is negative and the determinant of **J** is positive.

The local instability analysis of the more general economy presented in Section 5 also boils down to examining a dynamic system like (16), of which the Jacobian matrix **J** however is too complex to admit transparent analytical results. In the end of this appendix, we outline the elements of **J** on which our numerical exercises in Section 5 are based.

**Proof of Proposition 1.** In this case (16) is obtained by expressing $y_t$, $c_t$, $r_t$, and $u_t$ (along with $\tau_t^l$, $w_t$, $l_t$, and $R_t$) in terms of $k_t$ and $\lambda_t$, using the linearized versions of (5), (6), (8), (9), (11), (12), (13), and (14), and then substituting the outcomes into the linearized versions of (7) and (15).

The elements of the Jacobian matrix **J** are given by,

$$J_{11} = \frac{\rho(1-\alpha)}{s}\frac{1-\tau^l}{\tau^l - \underline{\tau}_b^{VCU}}, \quad J_{12} = \frac{\lambda}{y}\frac{(1-\alpha)(\rho+\delta)\rho}{\alpha s}\frac{\tau^l}{\tau^l - \underline{\tau}_b^{VCU}},$$

$$J_{21} = \frac{y}{\lambda}\left[s + \rho s_b - (1-\alpha)\tau^l - \frac{(1-\alpha)\left(1-\tau^l\right)}{\tau^l - \underline{\tau}_b^{VCU}}\right], \quad J_{22} = \rho - \frac{(1-\alpha)(\rho+\delta)}{\alpha}\frac{\underline{\tau}_b^{VCU}}{\tau^l - \underline{\tau}_b^{VCU}}.$$

The trace and determinant of **J** are then obtained as,

$$\mathcal{T}_b^{VCU} = -\underbrace{\left[\frac{\alpha\rho}{\rho+(1-\alpha)\delta}\left(1-\tau^l\right) + \frac{\rho+\delta}{\alpha}s_b\right]}_{+}\frac{\rho}{\tau^l - \underline{\tau}_b^{VCU}},$$

$$\mathcal{D}_b^{VCU} = \frac{1-\alpha}{\alpha}\frac{\rho(\rho+\delta)^2}{\rho+(1-\alpha)\delta}\frac{\Omega_b}{\tau^l - \underline{\tau}_b^{VCU}},$$

where recall that $\Omega_b$, as defined by the left side of (17), is a convex function of $\tau^l$. We can verify that $\Omega_b(0) = s > 0$ and $\Omega_b(1) = -\rho\left(\frac{\alpha}{\rho+\delta} + s_b\right) < 0$. These together confirm our claim in the text that $\Omega_b(\tau^l)$ has a unique root in $(0,1)$ (denoted by $\overline{\tau}_b$ in the text) and is strictly decreasing in $\tau^l$ for all $\tau^l \in [0,1]$.

Now it is clear that $\mathcal{T}_b^{VCU} < 0$ if and only if $\underline{\tau}_b^{VCU} < \tau^l$. Conditional on $\underline{\tau}_b^{VCU} < \tau^l$, then $\mathcal{D}_b^{VCU} > 0$ if and only if $\Omega_b > 0$, or, $\tau^l < \overline{\tau}_b$ (note that we restrict attention to labor income tax rates not exceeding 1). This establishes Proposition 1.

**Q.E.D.**

**Proof of Proposition 2.** We shall first note that this economy has the same steady state as the one studied in Proposition 1. Using a similar substitution technique as in proving Proposition 1 above, we obtain (16) for this case, with the elements of the Jacobian matrix $\mathbf{J}$ given by,

$$J_{11} = (\rho+\delta)(1-\alpha)\frac{1-\tau^l}{\tau^l-\underline{\tau}_b^{CCU}},\quad J_{12} = \frac{\lambda}{y}\frac{(1-\alpha)(\rho+\delta)^2}{\alpha}\frac{\tau^l}{\tau^l-\underline{\tau}_b^{CCU}},$$

$$J_{21} = \frac{y}{\lambda}\left[s+\rho s_b-(1-\alpha)\tau^l-\frac{(1-\alpha)\left(1-\tau^l\right)}{\tau^l-\underline{\tau}_b^{CCU}}\right],\ J_{22} = \rho-\frac{(1-\alpha)(\rho+\delta)}{\alpha}\frac{\underline{\tau}_b^{CCU}}{\tau^l-\underline{\tau}_b^{CCU}}.$$

The trace and determinant of $\mathbf{J}$ are then obtained as,

$$\mathcal{T}_b^{CCU} = -\frac{\overbrace{\delta(1-\alpha)\tau^l+\alpha\rho\left(1-\tau^l\right)+(\rho+\delta)s_b\left[\rho+(1-\alpha)\frac{\rho+\delta}{\alpha}\right]}^{+}}{\tau^l-\underline{\tau}_b^{CCU}},$$

$$\mathcal{D}_b^{CCU} = \frac{1-\alpha}{\alpha}(\rho+\delta)^2\frac{\Omega_b}{\tau^l-\underline{\tau}_b^{CCU}},$$

where recall that $\Omega_b$ is the same as the one given in the proof of Proposition 1 above.

Now we can see that $\mathcal{T}_b^{CCU}<0$ if and only if $\underline{\tau}_b^{CCU}<\tau^l$. Conditional on $\underline{\tau}_b^{CCU}<\tau^l$, then $\mathcal{D}_b^{CCU}>0$ if and only if $\Omega_b>0$, or, $\tau^l<\overline{\tau}_b$ (once again we restrict attention to labor income tax rates not exceeding 1). This proves Proposition 2.

**Q.E.D.**

**Proof of Proposition 3.** The steady state in this case differs from the one studied in Propositions 1 and 2. Yet we can use a similar substitution technique as in proving the above propositions to obtain (16) for this case, with the elements of the Jacobian matrix $\mathbf{J}$ given by,

$$J_{11} = -\frac{(1-\alpha)\rho}{s}\varepsilon_{11}^{VCU},\quad J_{12} = -\frac{\lambda}{y}\frac{(1-\alpha)(\rho+\delta)\rho}{\alpha s}\left(\varepsilon_{12}^{VCU}-1\right),$$

$$J_{21} = \frac{y}{\lambda}\left[(1-\alpha)\varepsilon_{11}^{VCU}-\frac{s-(1-\alpha)\tau^l}{1+\tau^c}\varepsilon_{21}^{VCU}\right],\quad J_{22} = \rho+\frac{\rho+\delta}{\alpha}\left[(1-\alpha)\varepsilon_{12}^{VCU}-\frac{s-(1-\alpha)\tau^l}{1+\tau^c}\varepsilon_{22}^{VCU}\right],$$

with the four auxiliary notations introduced as follows to help simplify exposition,

$$\varepsilon_{11}^{VCU} \equiv -\frac{1-\tau^l+\frac{s-(1-\alpha)\tau^l}{(1-\alpha)(1+\tau^c)}\left[\frac{\left(1-\tau^l\right)\tau^c}{(1+\tau^c)\tau^l}-\tau^c\right]}{\frac{\alpha\rho}{\rho+(1-\alpha)\delta}\Pi_c^{VCU}}\tau^l,\quad \varepsilon_{12}^{VCU} \equiv 1-\frac{\left(\tau^l\right)^2}{\frac{\alpha\rho}{\rho+(1-\alpha)\delta}\Pi_c^{VCU}},$$

$$\varepsilon_{21}^{VCU} \equiv -1 - \frac{\tau^c\left(1-\tau^l\right)}{\left(1+\tau^c\right)\tau^l}\left[1 - \frac{\alpha\rho}{\rho+(1-\alpha)\delta}\varepsilon_{11}^{VCU}\right], \quad \varepsilon_{22}^{VCU} \equiv -\frac{\tau^c\left(1-\tau^l\right)}{\left(1+\tau^c\right)\tau^l}\frac{\alpha\rho\left(1-\varepsilon_{12}^{VCU}\right)}{\rho+(1-\alpha)\delta},$$

where recall that $\Pi_c^{VCU}$, as defined by the left side of (31), is a convex function of $\tau^l$.

With some algebra, we obtain the trace and determinant of $\mathbf{J}$ as follows,

$$\mathcal{T}_c^{VCU} = -\frac{\overbrace{\rho\left(1-\tau^l\right) + \frac{s-(1-\alpha)\tau^l}{1+\tau^c}\frac{\tau^c}{1+\tau^c}\left[\frac{\rho+\delta}{\alpha}\left(\tau^c+\tau^l\right) + \frac{\rho}{1-\alpha}\frac{1-\tau^l}{\tau^l}\right]}^{+}}{\Pi_c^{VCU}}\tau^l,$$

$$\mathcal{D}_c^{VCU} = \left(\frac{\rho+\delta}{\alpha}\right)^2(1-\alpha)\,\tau^l\frac{\Omega_c}{\Pi_c^{VCU}},$$

where recall that $\Omega_c$ is given by the left side of (30), viewed as a function of $\tau^l$. With some algebra, we establish the following three properties: as $\tau^l$ approaches 0 from the right, $\Omega_c$ approaches $+\infty$; $\Omega_c(1) = -\frac{s-(1-\alpha)\tau^l}{1+\tau^c}\left(\frac{s\tau^c}{1-\alpha}+1\right) < 0$; and,

$$\Omega_c'(\tau^l) = -\frac{s\tau^c}{(1+\tau^c)^2}\frac{1-\tau^l}{\tau^l} - \frac{1-\alpha}{1+\tau^c}\overbrace{\left(\frac{s}{1-\alpha} - \frac{\tau^c+\tau^l}{1+\tau^c}\right)}^{+} - \frac{s-(1-\alpha)\tau^l}{(1+\tau^c)^2}\left[\frac{s}{1-\alpha}\frac{\tau^c}{\left(\tau^l\right)^2}+1\right] < 0, \text{ if } \tau^l \in (0,1].$$

These together confirm our claim in the text that $\Omega_c(\tau^l)$ has a unique root in $(0,1)$ (denoted by $\overline{\tau}_c$ in the text) and is strictly decreasing in $\tau^l$ for all $\tau^l \in (0,1]$.

On the other hand, we can verify that $\Pi_c^{VCU}$ viewed as a convex function of $\tau^l$ has the following properties: $\Pi_c^{VCU}(0) = -\frac{s\tau^c}{(1-\alpha)(1+\tau^c)^2} < 0$ and $\Pi_c^{VCU}(1) = \frac{(1-\alpha)(\rho+\delta)}{\alpha\rho} > 0$. These together confirm our claim in the text that $\Pi_c^{VCU}(\tau^l)$ has a negative root and a root in $(0,1)$ (denoted by $\underline{\tau}_c^{VCU}$ in the text) and is strictly increasing in $\tau^l$ for all $\tau^l \in [\underline{\tau}_c^{VCU}, 1]$.

Now it is straightforward to show that $\mathcal{T}_c^{VCU} < 0$ if and only if $\Pi_c^{VCU} > 0$, or, $\underline{\tau}_c^{VCU} < \tau^l$ (note that we restrict attention to nonnegative labor income tax rates). Conditional on $\Pi_c^{VCU} > 0$, then $\mathcal{D}_c^{VCU} > 0$ if and only if $\Omega_c > 0$, or, $\tau^l < \overline{\tau}_c$ (recall that we restrict attention to labor income tax rates not exceeding 1). This establishes Proposition 3.

**Q.E.D.**

**Proof of Proposition 4.** This economy has the same steady state as the one studied in Proposition 3. Using a similar substitution approach as used above, we obtain (16) for this case, with the elements of the Jacobian matrix $\mathbf{J}$ given by,

$$J_{11} = -(\rho+\delta)(1-\alpha)\,\varepsilon_{11}^{CCU}, \quad J_{12} = -\frac{\lambda}{y}\frac{(1-\alpha)(\rho+\delta)^2}{\alpha}\left(\varepsilon_{12}^{CCU}-1\right),$$

$$J_{21}=\frac{y}{\lambda}\left[(1-\alpha)\,\varepsilon_{11}^{CCU}-\frac{s-(1-\alpha)\,\tau^l}{1+\tau^c}\varepsilon_{21}^{CCU}\right],\quad J_{22}=\rho+\frac{\rho+\delta}{\alpha}\left[(1-\alpha)\,\varepsilon_{12}^{CCU}-\frac{s-(1-\alpha)\,\tau^l}{1+\tau^c}\varepsilon_{22}^{CCU}\right],$$

with the four auxiliary notations introduced as follows to help simplify exposition,

$$\varepsilon_{11}^{CCU}\equiv-\frac{1-\tau^l+\frac{s-(1-\alpha)\tau^l}{(1-\alpha)(1+\tau^c)}\left[\frac{\left(1-\tau^l\right)\tau^c}{(1+\tau^c)\tau^l}-\tau^c\right]}{\alpha\Pi_c^{CCU}}\tau^l,\quad \varepsilon_{12}^{CCU}\equiv1-\frac{\left(\tau^l\right)^2}{\alpha\Pi_c^{CCU}},$$

$$\varepsilon_{21}^{CCU}\equiv-1-\frac{\tau^c\left(1-\tau^l\right)}{\left(1+\tau^c\right)\tau^l}\left(1-\alpha\varepsilon_{11}^{CCU}\right),\quad \varepsilon_{22}^{CCU}\equiv-\frac{\tau^c\left(1-\tau^l\right)}{\left(1+\tau^c\right)\tau^l}\alpha\left(1-\varepsilon_{12}^{CCU}\right),$$

where recall that $\Pi_c^{CCU}$, as defined by the left side of (32), is a convex function of $\tau^l$.

The trace and determinant of $\mathbf{J}$ are then obtained as follows,

$$\mathcal{T}_c^{CCU}=-\frac{\overbrace{\rho\left(1-\tau^l\right)+\frac{1-\alpha}{\alpha}\delta\tau^l+\frac{s-(1-\alpha)\,\tau^l}{1+\tau^c}\frac{\tau^c}{1+\tau^c}\left[\frac{\left(\tau^c+\tau^l\right)(\rho+\delta)}{\alpha}+\frac{\rho}{1-\alpha}\frac{1-\tau^l}{\tau^l}\right]}^{+}}{\Pi_c^{CCU}}\tau^l,$$

$$\mathcal{D}_c^{CCU}=\left(\frac{\rho+\delta}{\alpha}\right)^2(1-\alpha)\,\tau^l\frac{\Omega_c}{\Pi_c^{CCU}},$$

where recall that $\Omega_c$ is the same as the one given in the proof of Proposition 3 above.

It is easy to verify that $\Pi_c^{CCU}$ viewed as a convex function of $\tau^l$ has the following properties: $\Pi_c^{CCU}(0)=-\frac{s\tau^c}{(1-\alpha)(1+\tau^c)^2}<0$ and $\Pi_c^{CCU}(1)=\frac{1-\alpha}{\alpha}>0$. These together confirm our claim in the text that $\Pi_c^{CCU}(\tau^l)$ has a negative root and a root in $(0,1)$ (denoted by $\underline{\tau}_c^{CCU}$ in the text) and is strictly increasing in $\tau^l$ for all $\tau^l\in[\underline{\tau}_c^{CCU},1]$.

We can now show that $\mathcal{T}_c^{CCU}<0$ if and only if $\Pi_c^{CCU}>0$, or, $\underline{\tau}_c^{CCU}<\tau^l$ (recall that we restrict attention to nonnegative labor income tax rates). Given that $\Pi_c^{CCU}>0$, then $\mathcal{D}_c^{CCU}>0$ if and only if $\Omega_c>0$, or, $\tau^l<\overline{\tau}_c$ (once again we restrict attention to labor income tax rates not exceeding 1). This establishes Proposition 4.

**Q.E.D.**

**Proof of Inequality (33).**

We have proved above that Eqs. (31) and (32) each has a negative root and a positive root, and recall that $\underline{\tau}_c^{VCU}$ and $\underline{\tau}_c^{CCU}$ are the two positive roots. To simplify exposition, let

$$a_0\equiv-\frac{s\tau^c}{(1-\alpha)(1+\tau^c)^2},\quad a_1\equiv\left(1+\frac{s}{1-\alpha}\right)\frac{\tau^c}{(1+\tau^c)^2}-1,$$

$$a_2^{VCU} \equiv \frac{\rho + (1-\alpha)\delta}{\alpha\rho} - \frac{\tau^c}{(1+\tau^c)^2}, \quad a_2^{CCU} \equiv \frac{1}{\alpha} - \frac{\tau^c}{(1+\tau^c)^2}. \tag{A.1}$$

We then have

$$\underline{\tau}_c^{VCU} = \frac{-2a_0}{a_1 + \sqrt{a_1^2 - 4a_0 a_2^{VCU}}} < \frac{-2a_0}{a_1 + \sqrt{a_1^2 - 4a_0 a_2^{CCU}}} = \underline{\tau}_c^{CCU},$$

where the inequality holds because $a_0 < 0$ and $a_2^{VCU} > a_2^{CCU} > 0$. This establishes (33).

**Q.E.D.**

**Proof of Proposition 5.** The steady state in this case generalizes those studied in Propositions 1-4. Yet we can still apply a similar substitution approach used above to obtain (16) for this general case, with the elements of the Jacobian matrix $\mathbf{J}$ given by,

$$J_{11} = -\frac{(1-\alpha)\,\rho}{s}\tilde{\varepsilon}_{11}^{VCU}, \quad J_{12} = -\frac{\lambda}{y}\frac{(1-\alpha)\,(\rho+\delta)\,\rho}{\alpha s}\left(\tilde{\varepsilon}_{12}^{VCU} - 1\right),$$

$$J_{21} = \frac{y}{\lambda}\left[(1-\alpha)\,\tilde{\varepsilon}_{11}^{VCU} - \tfrac{s+\rho s_b-(1-\alpha)\tau^l}{1+\tau^c}\tilde{\varepsilon}_{21}^{VCU}\right], \quad J_{22} = \rho + \frac{\rho+\delta}{\alpha}\left[(1-\alpha)\,\tilde{\varepsilon}_{12}^{VCU} - \tfrac{s+\rho s_b-(1-\alpha)\tau^l}{1+\tau^c}\tilde{\varepsilon}_{22}^{VCU}\right],$$

with the four auxiliary notations introduced as follows to help simplify exposition,

$$\tilde{\varepsilon}_{11}^{VCU} = -\frac{1-\tau^l + \frac{1}{1-\alpha}\frac{s+\rho s_b-(1-\alpha)\tau^l}{1+\tau^c}\tau^c\left[\frac{1-\tau^l}{(1+\tau^c)\tau^l} - 1\right]}{\frac{\alpha\rho}{\rho+(1-\alpha)\delta}\Pi_{b,c}^{VCU}}\tau^l, \quad \tilde{\varepsilon}_{12}^{VCU} = 1 - \frac{\left(\tau^l\right)^2}{\frac{\alpha\rho}{\rho+(1-\alpha)\delta}\Pi_{b,c}^{VCU}},$$

$$\tilde{\varepsilon}_{21}^{VCU} = -1 - \frac{\tau^c\left(1-\tau^l\right)}{(1+\tau^c)\,\tau^l}\left[1 - \frac{\alpha\rho}{\rho+(1-\alpha)\,\delta}\tilde{\varepsilon}_{11}^{VCU}\right], \quad \tilde{\varepsilon}_{22}^{VCU} = -\frac{\tau^c\left(1-\tau^l\right)}{(1+\tau^c)\,\tau^l}\frac{\alpha\rho}{\rho+(1-\alpha)\,\delta}\left(1-\tilde{\varepsilon}_{12}^{VCU}\right),$$

where recall that $\Pi_{b,c}^{VCU}$, as defined by the left side of (38), is a convex function of $\tau^l$.

With some algebra, we obtain the trace and determinant of $\mathbf{J}$ as follows,

$$\mathcal{T}_{b,c}^{VCU} = -\frac{\overbrace{\rho\left(1-\tau^l\right) + \left(\frac{\rho+\delta}{\alpha}\right)^2 s_b s + \frac{s+\rho s_b-(1-\alpha)\tau^l}{1+\tau^c}\frac{\tau^c}{1+\tau^c}\left[\frac{(\rho+\delta)\left(\tau^c+\tau^l\right)}{\alpha} + \frac{\rho}{1-\alpha}\frac{1-\tau^l}{\tau^l}\right]}^{+}}{\Pi_{b,c}^{VCU}}\tau^l,$$

$$\mathcal{D}_{b,c}^{VCU} = \left(\frac{\rho+\delta}{\alpha}\right)^2 (1-\alpha)\,\tau^l\frac{\Omega_{b,c}}{\Pi_{b,c}^{VCU}},$$

where recall that $\Omega_{b,c}$ is given by the left side of (37), viewed as a function of $\tau^l$. With some algebra, we establish the following three properties: as $\tau^l$ approaches 0 from the right, $\Omega_{b,c}$ approaches $+\infty$; $\Omega_{b,c}(1) = -\frac{s+\rho s_b-(1-\alpha)\tau^l}{1+\tau^c}\left(\frac{s\tau^c}{1-\alpha}+1\right) < 0$; and,

$$\Omega_{b,c}'(\tau^l) = -\frac{s\tau^c}{(1+\tau^c)^2}\frac{1-\tau^l}{\tau^l} - \frac{1-\alpha}{1+\tau^c}\overbrace{\left(\frac{s}{1-\alpha} - \frac{\tau^c+\tau^l}{1+\tau^c}\right)}^{+} - \frac{s+\rho s_b-(1-\alpha)\tau^l}{(1+\tau^c)^2}\left[\frac{s}{1-\alpha}\frac{\tau^c}{\left(\tau^l\right)^2}+1\right] < 0, \text{ if } \tau^l \in (0,1].$$

These together confirm our claim in the text that $\Omega_{b,c}(\tau^l)$ has a unique root in $(0,1)$ (denoted by $\overline{\tau}_{b,c}$ in the text) and is strictly decreasing in $\tau^l$ for all $\tau^l \in (0,1]$.

On the other hand, the fact that $\Pi_{b,c}^{VCU}(0) = -\frac{(s+\rho s_b)\tau^c}{(1-\alpha)(1+\tau^c)^2} < 0$ confirms our claim in the text that $\Pi_{b,c}^{VCU}(\tau^l)$ as a convex function of $\tau^l$ has a negative root and a positive root (denoted by $\underline{\tau}_{b,c}^{VCU}$ in the text) and is strictly increasing in $\tau^l$ for all $\tau^l \in [\underline{\tau}_{b,c}^{VCU}, +\infty)$.

Now it is straightforward to show that $\mathcal{T}_{b,c}^{VCU} < 0$ if and only if $\Pi_{b,c}^{VCU} > 0$, or, $\underline{\tau}_{b,c}^{VCU} < \tau^l$ (note that we restrict attention to nonnegative labor income tax rates). Conditional on $\Pi_{b,c}^{VCU} > 0$, then $\mathcal{D}_{b,c}^{VCU} > 0$ if and only if $\Omega_{b,c} > 0$, or, $\tau^l < \overline{\tau}_{b,c}$ (recall that we restrict attention to labor income tax rates not exceeding 1). This establishes Proposition 5.

**Q.E.D.**

**Proof of Proposition 6.** This economy has the same steady state as the one studied in Proposition 5. Using a similar substitution approach as used above, we obtain (16) for this case, with the elements of the Jacobian matrix $\mathbf{J}$ given by,

$$J_{11} = -\left(\rho+\delta\right)\left(1-\alpha\right)\tilde{\varepsilon}_{11}^{CCU}, \quad J_{12} = -\frac{\lambda}{y}\frac{\left(1-\alpha\right)\left(\rho+\delta\right)^2}{\alpha}\left(\tilde{\varepsilon}_{12}^{CCU}-1\right),$$

$$J_{21} = \frac{y}{\lambda}\left[\left(1-\alpha\right)\tilde{\varepsilon}_{11}^{CCU} - \tfrac{s+\rho s_b-(1-\alpha)\tau^l}{1+\tau^c}\tilde{\varepsilon}_{21}^{CCU}\right], \quad J_{22} = \frac{\rho+\delta}{\alpha}\left[\left(1-\alpha\right)\tilde{\varepsilon}_{12}^{CCU} - \tfrac{s+\rho s_b-(1-\alpha)\tau^l}{1+\tau^c}\tilde{\varepsilon}_{22}^{CCU}\right]+\rho,$$

with the four auxiliary notations introduced as follows to help simplify exposition,

$$\tilde{\varepsilon}_{11}^{CCU} = -\frac{1-\tau^l+\frac{1}{1-\alpha}\frac{s+\rho s_b-(1-\alpha)\tau^l}{1+\tau^c}\tau^c\left[\frac{1-\tau^l}{(1+\tau^c)\tau^l}-1\right]}{\alpha\Pi_{b,c}^{CCU}}\tau^l, \quad \tilde{\varepsilon}_{12}^{CCU} = 1-\frac{\left(\tau^l\right)^2}{\alpha\Pi_{b,c}^{CCU}},$$

$$\tilde{\varepsilon}_{21}^{CCU} = -1-\frac{\tau^c\left(1-\tau^l\right)}{\left(1+\tau^c\right)\tau^l}\left(1-\alpha\tilde{\varepsilon}_{11}^{CCU}\right), \quad \tilde{\varepsilon}_{22}^{CCU} = -\frac{\tau^c\left(1-\tau^l\right)}{\left(1+\tau^c\right)\tau^l}\alpha\left(1-\tilde{\varepsilon}_{12}^{CCU}\right),$$

where recall that $\Pi_{b,c}^{CCU}$, as defined by the left side of (39), is a convex function of $\tau^l$.

The trace and determinant of $\mathbf{J}$ are then obtained as follows,

$$\mathcal{T}_{b,c}^{CCU} = -\frac{\overbrace{\rho\left(1-\tau^l\right)+\frac{1-\alpha}{\alpha}\delta\tau^l+\left(\frac{\rho+\delta}{\alpha}\right)^2 s_b s+\frac{s+\rho s_b-(1-\alpha)\tau^l}{1+\tau^c}\frac{\tau^c}{1+\tau^c}\left[\frac{\left(\tau^c+\tau^l\right)\left(\rho+\delta\right)}{\alpha}+\frac{\rho}{1-\alpha}\frac{1-\tau^l}{\tau^l}\right]}^{+}}{\Pi_{b,c}^{CCU}}\tau^l,$$

$$\mathcal{D}_{b,c}^{CCU} = \left(\frac{\rho+\delta}{\alpha}\right)^2\left(1-\alpha\right)\tau^l\frac{\Omega_{b,c}}{\Pi_{b,c}^{CCU}},$$

where recall that $\Omega_{b,c}$ is the same as the one given in the proof of Proposition 5 above.

On the other hand, the fact that $\Pi_{b,c}^{CCU}(0) = -\frac{(s+\rho s_b)\tau^c}{(1-\alpha)(1+\tau^c)^2} < 0$ confirms our claim in the text that $\Pi_{b,c}^{CCU}(\tau^l)$ as a convex function of $\tau^l$ has a negative root and a positive root (denoted by $\underline{\tau}_{b,c}^{CCU}$ in the text) and is strictly increasing in $\tau^l$ for all $\tau^l \in [\underline{\tau}_{b,c}^{CCU}, +\infty)$.

We can now show that $\mathcal{T}_{b,c}^{CCU} < 0$ if and only if $\Pi_{b,c}^{CCU} > 0$, or, $\underline{\tau}_{b,c}^{CCU} < \tau^l$ (recall that we restrict attention to nonnegative labor income tax rates). Given that $\Pi_{b,c}^{CCU} > 0$, then $\mathcal{D}_{b,c}^{CCU} > 0$ if and only if $\Omega_{b,c} > 0$, or, $\tau^l < \overline{\tau}_{b,c}$ (once again we restrict attention to labor income tax rates not exceeding 1). This establishes Proposition 6.

**Q.E.D.**

**Proof of Inequality (40).**

We have proved above that Eqs. (38) and (39) each has a negative root and a positive root, and recall that $\underline{\tau}_{b,c}^{VCU}$ and $\underline{\tau}_{b,c}^{CCU}$ are the two positive roots. To simplify exposition, let

$$b_0 \equiv -\frac{(s+\rho s_b)\tau^c}{(1-\alpha)(1+\tau^c)^2}, \quad b_1 \equiv \left(1 + \frac{s}{1-\alpha}\right)\frac{\tau^c}{(1+\tau^c)^2} + \left[\frac{\rho\tau^c}{(1-\alpha)(1+\tau^c)^2} - \frac{\rho+\delta}{\alpha}\right] s_b - 1.$$

We then have

$$\underline{\tau}_{b,c}^{VCU} = \frac{-2b_0}{b_1 + \sqrt{b_1^2 - 4b_0 a_2^{VCU}}} < \frac{-2b_0}{b_1 + \sqrt{b_1^2 - 4b_0 a_2^{CCU}}} = \underline{\tau}_{b,c}^{CCU},$$

where $a_2^{VCU}$ and $a_2^{CCU}$ are as defined in (A.1), and where the inequality holds because $b_0 < 0$ and $a_2^{VCU} > a_2^{CCU} > 0$. This establishes (40).

**Q.E.D.**

**The dynamic systems in Section 5.** In both the case with variable capacity utilization and the case with constant capacity utilization, local instability analysis boils down to analyzing a dynamic system like (16). To help simplify exposition, we introduce the following auxiliary notations,

$$\epsilon_c \equiv \frac{\tau^c \frac{c}{y}}{(1-\alpha)\tau^l + \alpha\tau^k - (\rho+\delta)s_b}, \quad \epsilon_u \equiv \frac{s_b(1-\tau^k)\alpha\frac{y}{k}}{(1-\alpha)\tau^l + \alpha\tau^k - (\rho+\delta)s_b},$$

$$\epsilon_k \equiv \frac{(1-\tau^k)\alpha\frac{y}{k}s_b}{(1-\alpha)\tau^l + \alpha\tau^k - (\rho+\delta)s_b}, \quad \epsilon_\tau \equiv \frac{(1-\alpha)\tau^l + \alpha\tau^k + \tau^c\frac{c}{y} + \alpha\frac{y}{k}\tau^k s_b}{(1-\alpha)\tau^l + \alpha\tau^k - (\rho+\delta)s_b},$$

where

$$\frac{y}{k} = \frac{1}{\alpha}\frac{\rho+\delta}{1-\tau^k}, \quad \frac{c}{y} = \frac{(1-\alpha)(1-\tau^l) + \rho\left[\frac{\alpha(1-\tau^k)}{\rho+\delta} + s_b\right]}{1+\tau^c}.$$

The Jacobian matrix **J** differers across the cases with variable and constant capacity utilization.

*Under variable capacity utilization*

The elements of **J** under variable capacity utilization are given by,

$$J_{11} = \alpha \tfrac{y}{k} \left[ \left( 1 - \tau^k \right) \left( u_\lambda^{VCU} - y_\lambda^{VCU} \right) + \tau^k \tau_\lambda^{VCU} \right], \ J_{12} = \alpha \left( \tfrac{y}{k} \right)^2 \tfrac{\lambda}{y} \left[ \left( 1 - \tau^k \right) \left( 1 + u_k^{VCU} - y_k^{VCU} \right) + \tau^k \tau_k^{VCU} \right],$$

$$J_{21} = \tfrac{y}{\lambda} \left[ y_\lambda^{VCU} - \alpha \left( 1 - \tau^k \right) u_\lambda^{VCU} + \tfrac{c}{y} \left( 1 + \tfrac{\tau^c \tau_\lambda^{VCU}}{1+\tau^c} \right) \right], \ J_{22} = \tfrac{y}{k} \left[ y_k^{VCU} - \alpha \left( 1 - \tau^k \right) u_k^{VCU} + \tfrac{c}{y} \tfrac{\tau^c \tau_k^{VCU}}{1+\tau^c} \right] - \delta,$$

with the additional auxiliary notations introduced as follows to further simplify exposition,

$$\tau_\lambda^{VCU} \equiv - \frac{(1-\alpha)(\theta + \epsilon_u) - \alpha(\theta - 1)\epsilon_c}{\alpha(\theta-1)\left( \epsilon_\tau - \frac{\tau^c}{1+\tau^c}\epsilon_c \right) - \alpha(1+\epsilon_u)\frac{\tau^k}{1-\tau^k} - (1-\alpha)(\theta+\epsilon_u)\frac{\tau^l}{1-\tau^l}},$$

$$\tau_k^{VCU} \equiv - \frac{\alpha(\theta-1)(1+\epsilon_k)}{\alpha(\theta-1)\left( \epsilon_\tau - \frac{\tau^c}{1+\tau^c}\epsilon_c \right) - \alpha(1+\epsilon_u)\frac{\tau^k}{1-\tau^k} - (1-\alpha)(\theta+\epsilon_u)\frac{\tau^l}{1-\tau^l}},$$

$$y_\lambda^{VCU} \equiv \left[ \epsilon_c \theta + \left( \frac{\tau^c}{1+\tau^c}\theta\epsilon_c - \theta\epsilon_\tau + \epsilon_u \frac{\tau^k}{1-\tau^k} \right) \tau_\lambda^{VCU} \right] \frac{1}{\theta + \epsilon_u},$$

$$y_k^{VCU} \equiv \left[ \epsilon_u - \epsilon_k \theta + \left( \frac{\tau^c}{1+\tau^c}\theta\epsilon_c - \theta\epsilon_\tau + \epsilon_u \frac{\tau^k}{1-\tau^k} \right) \tau_k^{VCU} \right] \frac{1}{\theta + \epsilon_u},$$

$$u_\lambda^{VCU} \equiv \frac{1}{\theta} \left( y_\lambda^{VCU} - \frac{\tau^k}{1-\tau^k} \tau_\lambda^{VCU} \right), \ u_k^{VCU} \equiv \frac{1}{\theta} \left( y_k^{VCU} - 1 - \frac{\tau^k}{1-\tau^k} \tau_k^{VCU} \right).$$

*Under constant capacity utilization*

The elements of **J** under constant capacity utilization are given by,

$$J_{11} = \alpha \frac{y}{k} \frac{\left(1-\tau^l\right)\tau^k\left(1-\alpha l_\lambda^{CCU}\right) - (1-\alpha)\left(1-\tau^k\right)\tau^l l_\lambda^{CCU}}{\tau^l}, \ J_{12} = \alpha \left( \frac{y}{k} \right)^2 \frac{\lambda}{y} \frac{(1-\alpha)\left(1-\tau^k\right)\tau^l + \left(1-\tau^l\right)\alpha\tau^k}{\tau^l} \left( 1 - l_k^{CCU} \right),$$

$$J_{21} = \frac{y}{\lambda} \left[ (1-\alpha) l_\lambda^{CCU} - \frac{c}{y} c_\lambda^{CCU} \right], \ J_{22} = \frac{y}{k} \left[ \alpha + (1-\alpha) l_k^{CCU} - \frac{c}{y} c_k^{CCU} \right] - \delta,$$

with the additional auxiliary notations introduced as follows to further simplify exposition,

$$l_\lambda^{CCU} \equiv \frac{\epsilon_c \tau^l - \left( \epsilon_\tau - \frac{\tau^c}{1+\tau^c}\epsilon_c \right)\left(1-\tau^l\right)}{(1-\alpha)\tau^l - \alpha\left( \epsilon_\tau - \frac{\tau^c}{1+\tau^c}\epsilon_c \right)\left(1-\tau^l\right)}, \quad l_k^{CCU} \equiv 1 - \frac{(1+\epsilon_k)\tau^l}{(1-\alpha)\tau^l - \alpha\left( \epsilon_\tau - \frac{\tau^c}{1+\tau^c}\epsilon_c \right)\left(1-\tau^l\right)},$$

$$c_\lambda^{CCU} \equiv -1 - \frac{\tau^c}{1+\tau^c}\frac{1-\tau^l}{\tau^l}\left(1 - \alpha l_\lambda^{CCU}\right), \ c_k^{CCU} \equiv - \frac{\tau^c}{1+\tau^c}\alpha\frac{1-\tau^l}{\tau^l}\left(1 - l_k^{CCU}\right).$$

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Table 1. Estimated effective tax rates[a] and debt-GDP ratio[b]

| | $(\tau^l, \tau^k, \tau^c)$ in 1988 | $(\tau^l, \tau^k, \tau^c)$ in 1996 | $s_b$ |
|---|---|---|---|
| US | $(0.28538, 0.3991, 0.05221)$ | $(0.27733, 0.3962, 0.05467)$ | 0.855 |
| UK | $(0.26729, 0.5872, 0.17243)$ | $(0.24406, 0.4717, 0.15245)$ | 0.642 |
| GE | $(0.41174, 0.2734, 0.14721)$ | $(0.42384, 0.2391, 0.16395)$ | 0.424 |
| CA | $(0.28017, 0.39, 0.13353)$ | $(0.32633, 0.5066, 0.10396)$ | 0.409 |
| JP | $(0.26571, 0.507, 0.05187)$ | $(0.27439, 0.4261, 0.06)$ | 1.429 |

[a]Estimated effective tax rates on factor incomes and consumption for 1988 are taken from Mendoza et al. (1994); updated estimates for 1996 are available at http://www.sas.upenn.edu/~egme/pp/newtaxdata.pdf.

[b]Data for 2014 from OECD Economic Outlook. Debt is measured by net financial liabilities held by the public.

Table 2. (In)Stability properties for different countries (1988,1996)

| | constant capacity utilization | | | | variable capacity utilization | | | |
|---|---|---|---|---|---|---|---|---|
| | $(\tau^l, s_b)$ | $(\tau^l, \tau^c)$ | $(\tau^l, s_b, \tau^c)$ | $(\tau^l, s_b, \tau^c, \tau^k)$ | $(\tau^l, s_b)$ | $(\tau^l, \tau^c)$ | $(\tau^l, s_b, \tau^c)$ | $(\tau^l, s_b, \tau^c, \tau^k)$ |
| US | (S,S) | (S,S) | (S,S) | (S,S) | (I,I) | (I,I) | (I,I) | (I,I) |
| UK | (S,S) | (S,S) | (S,S) | (S,S) | (I,I) | (I,I) | (I,I) | (I,I) |
| GE | (I,I) | (I,I) | (I,I) | (I,I) | (I,I) | (I,I) | (I,I) | (I,I) |
| CA | (S,S) | (S,S) | (S,S) | (S,S) | (I,I) | (I,I) | (I,I) | (I,I) |
| JP | (S,S) | (S,S) | (S,S) | (S,S) | (I,I) | (I,I) | (I,I) | (I,I) |

Notes: $(\tau^l, s_b)$ - balanced-budget rule with labor income taxes and public debt, $(\tau^l, \tau^c)$ - balanced-budget rule with labor income and consumption taxes, $(\tau^l, s_b, \tau^c)$ - balanced-budget rule with labor income and consumption taxes and public debt, $(\tau^l, s_b, \tau^c, \tau^k)$ - balanced-budget rule with labor and capital income as well as consumption taxes, along with public debt. S - stability, I - instability.

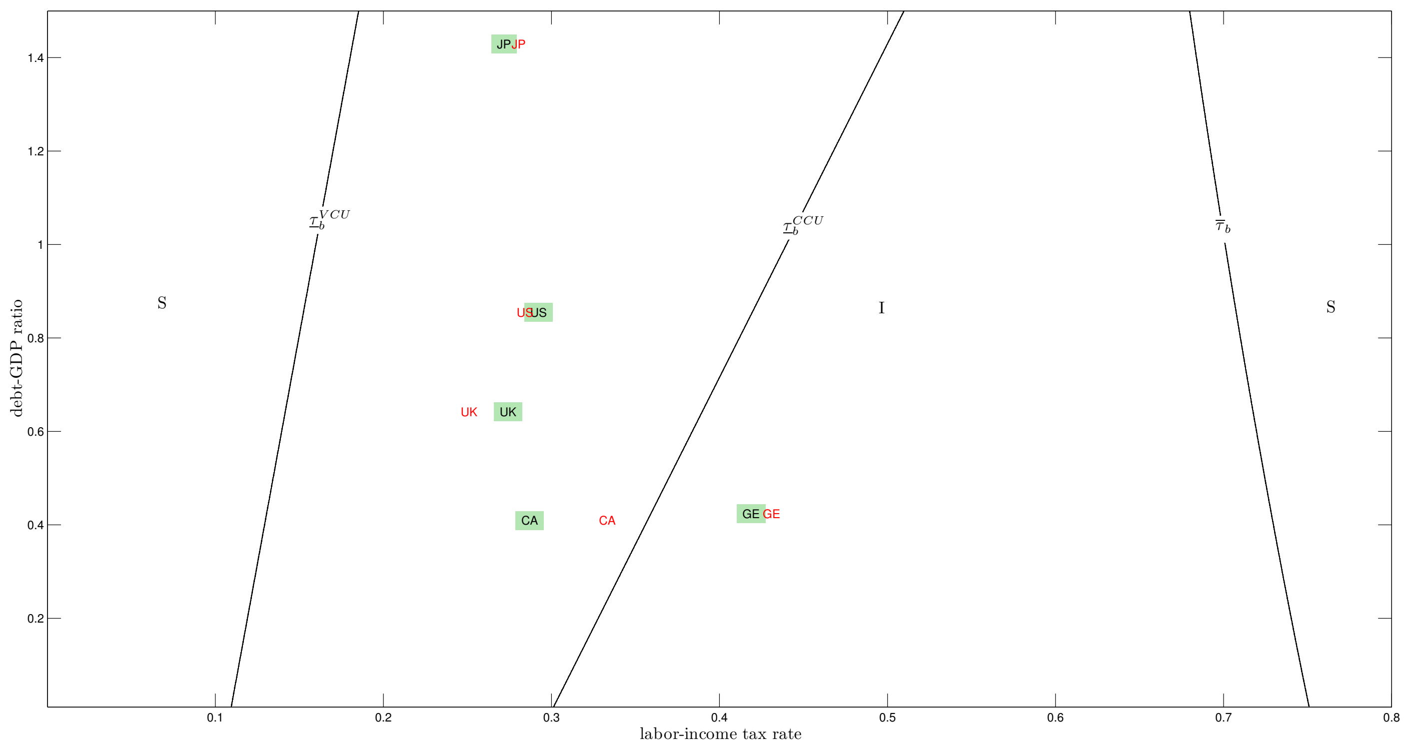


Figure 1. Balanced-budget rule with labor income taxes and public debt.

Shadowed (unshadowed) data are for 1988 (1996). S - stability, I - instability.

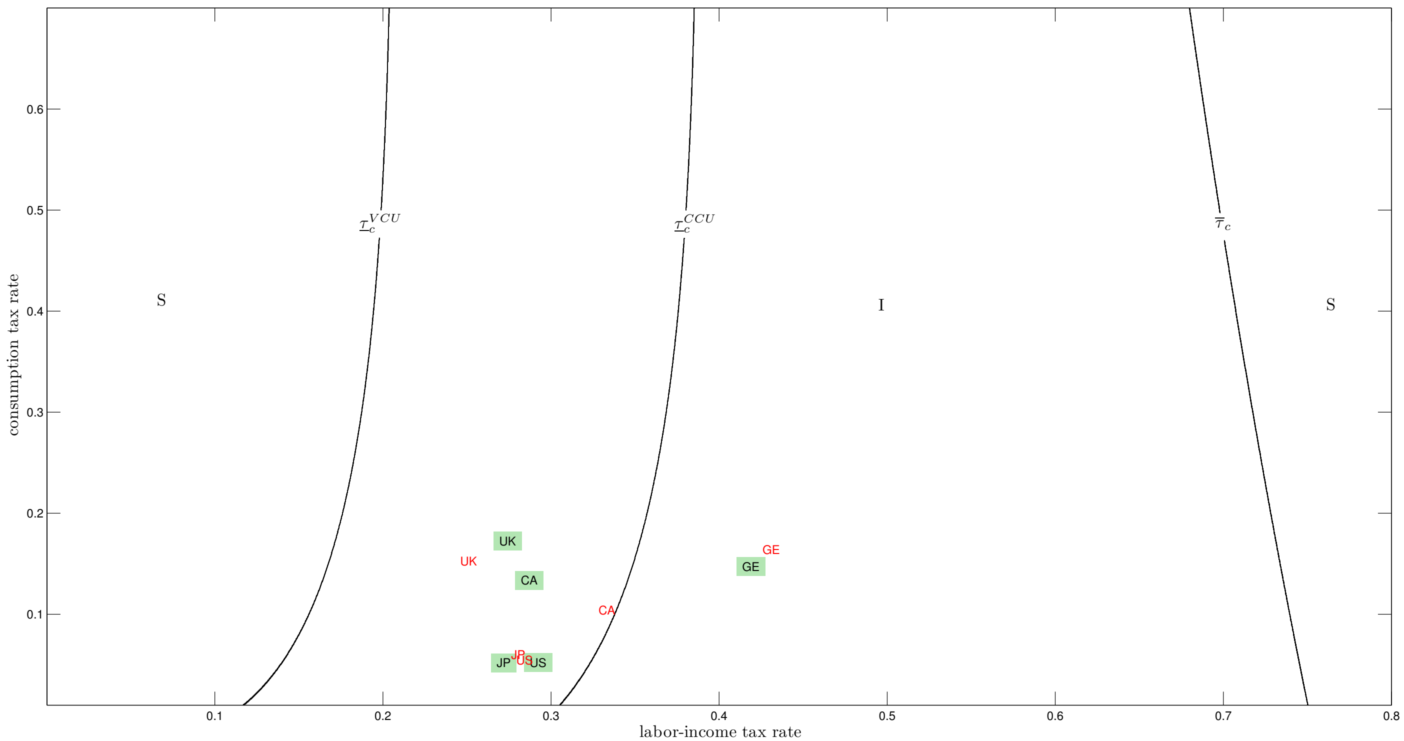


Figure 2. Balanced-budget rule with labor income and consumption taxes.

Shadowed (unshadowed) data are for 1988 (1996). S - stability, I - instability.

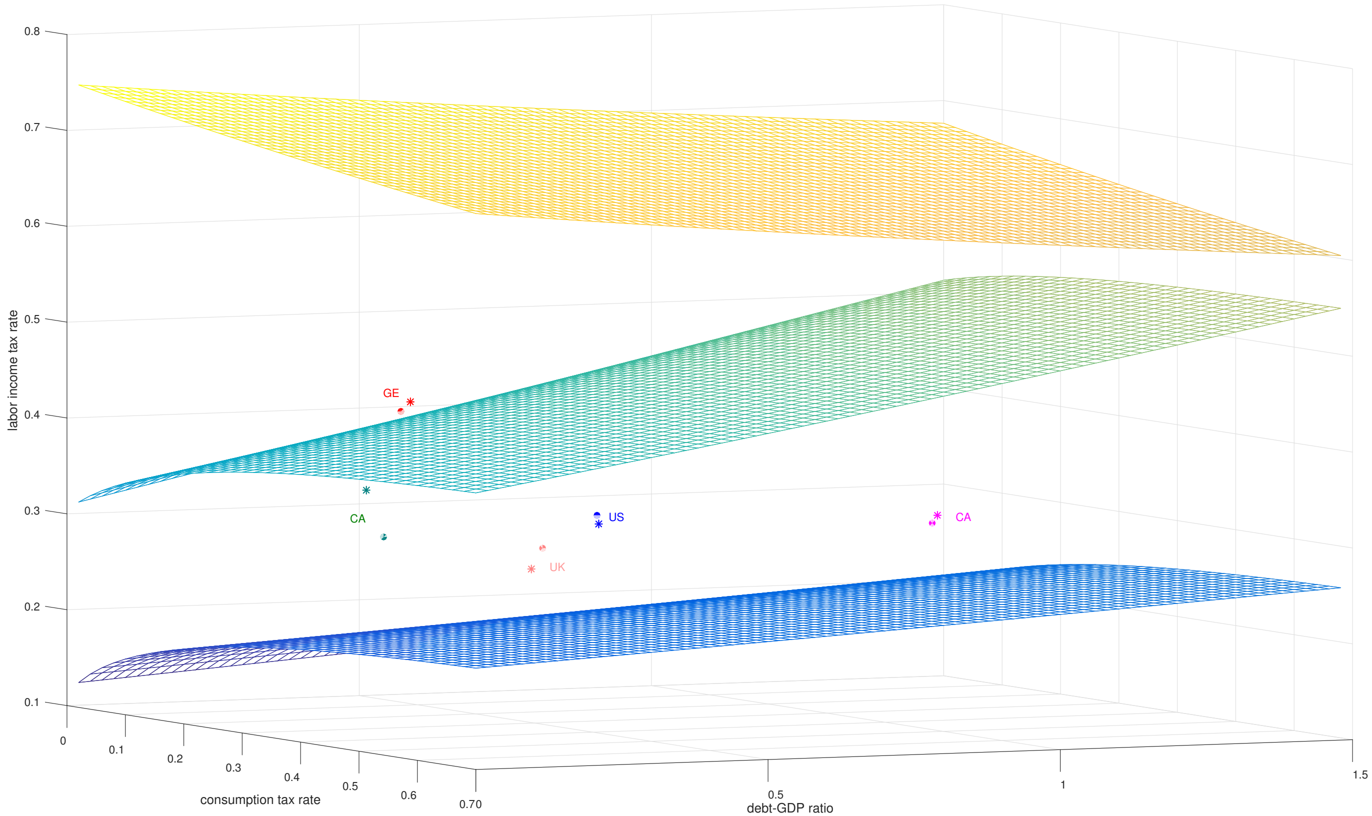


Figure 3. Balanced-budget rule with labor income and consumption taxes and public debt.

The space between the upper and lower (middle) surfaces is the instability region with variable (constant) capacity utilization, while the remaining space is the stability region. Data represented by "." ("*") are for 1988 (1996).

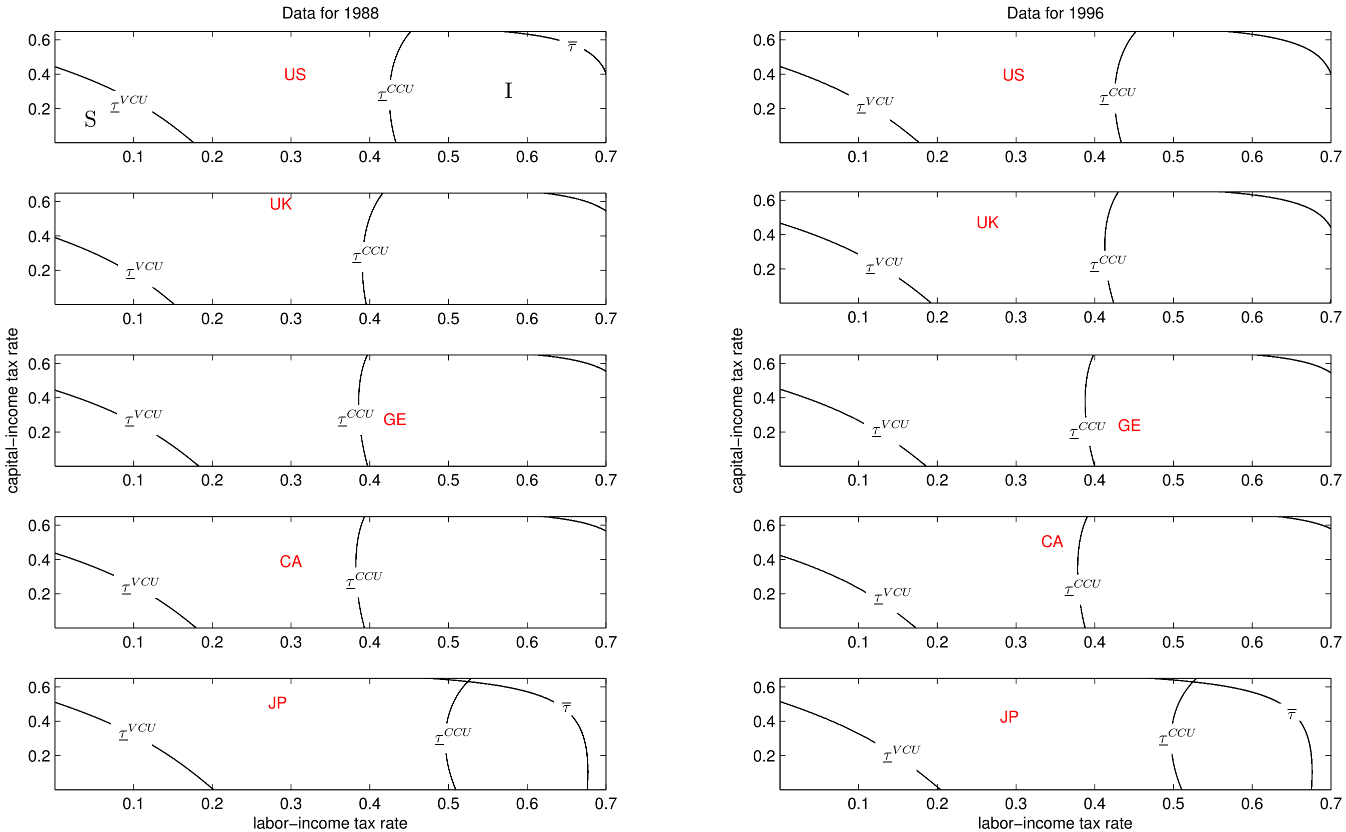


Figure 4. Balanced-budget rule with labor and capital income as well as consumption taxes, along with public debt. S - stability, I - instability.