Vanderbilt University Department of Economics Working Paper Series
Nashville, TN 37235-1819

Vanderbilt University Department of Economics Working Papers 16-00008

Existence of Share Equilibrium in Symmetric Local Public Good Economies*

Anne Van den nouweland
University of Oregon

Myrna Wooders
Vanderbilt University

Abstract

Share equilibrium was introduced in van den Nouweland and Wooders (2011) as an extension of ratio equilibrium to local public good economies. In that paper, we took an axiomatic approach to motivate share equilibrium. In the current paper we consider questions related to the existence of share equilibrium and we derive a necessary and su□cient condition for the existence of share equilibrium in symmetric economies. Along the way, we develop a deeper understanding of the possible variation in share equilibrium by considering when symmetric players necessarily have the same share indices in equilibrium.

We are grateful to Nizar Allouch, Nick Baigent, Jean-Marc Bonnisseau, Bernard Cornet, Shlomo Weber and other participants at several presentations of this work for stimulating comments. We are also grateful to the Douglas Grey Fund for Research in Economics at Vanderbilt for financial support.
Citation: Anne Van den nouweland and Myrna Wooders, (2016) "Existence of Share Equilibrium in Symmetric Local Public Good Economies*", *Vanderbilt University Department of Economics Working Papers*, VUECON-16-00008.
Contact: Anne Van den nouweland - annev@uoregon.edu, Myrna Wooders - myrna.wooders@gmail.com.
Submitted: March 12, 2016. **Published:** March 13, 2016.
URL:http://www.accessecon.com/Pubs/VUECON/VUECON-16-00008.pdf

Existence of Share Equilibrium in Symmetric Local Public Good Economies*

A. van den Nouweland[†] and M. Wooders[‡]

February 13, 2016

Abstract

Share equilibrium was introduced in van den Nouweland and Wooders (2011) as an extension of ratio equilibrium to local public good economies. In that paper, we took an axiomatic approach to motivate share equilibrium. In the current paper we consider questions related to the existence of share equilibrium and we derive a necessary and sufficient condition for the existence of share equilibrium in symmetric economies. Along the way, we develop a deeper understanding of the possible variation in share equilibrium by considering when symmetric players necessarily have the same share indices in equilibrium.

Keywords: Share equilibrium, Local public goods, Cost shares, Core, Top convexity

JEL codes: D7, C7

*We are grateful to Nizar Allouch, Nick Baigent, Jean-Marc Bonnisseau, Bernard Cornet, Shlomo Weber and other participants at several presentations of this work for stimulating comments. We are also grateful to the Douglas Grey Fund for Research in Economics at Vanderbilt University for financial support

[†]Corresponding author. Department of Economics, University of Oregon, Eugene, OR 97403-1285, USA. E-mail: annev@uoregon.edu. Phone: (+1) 541-346-1267. Fax: (+1) 541-346-1243.

[‡]Department of Economics, Vanderbilt University, Nashville, TN 37235, USA, myrna.wooders@vanderbilt.edu.

1 Introduction

Often public goods or services are excludable and thus provided only to members of groups (clubs, jurisdictions, coalitions, and so on). In many such situations, consumers pay cost shares rather than prices. For example, the share of homeowner association costs paid by a condominium owner may be determined by the relative size of the individual home owner's condominium. Dinner parties in restaurants often split the bill equally among the diners or use some formula for splitting the bill that takes into account the diners' relative standing. Equilibrium concepts for such situations are thus of interest. In a prior paper, the authors introduced the concept of *share equilibrium.* A share equilibrium specifies one share index for each individual; these indices represent the individuals' relative standings in groups and determine their cost shares in any jurisdiction they may feasibly join.[1] Van den Nouweland and Wooders (2011) show, via an axiomatization, that share equilibrium is an extension of Lindahl's original equilibrium concept (Lindahl 1919) to economies with a local public good and possibly multiple jurisdictions. The goal of the current paper is to look deeper into the concept of share equilibrium and provide some characterization results and an existence result for symmetric economies.

Our definition of an economy with local public goods is now standard in the literature. The preferences of an individual depend on the membership of the jurisdiction to which they belong and their consumption of a private good and a public good for that jurisdiction. Each individual belongs to exactly one jurisdiction.[2] Our first theorem demonstrates that share equilibrium outcomes are in the core of the economy. Among other things, this implies that share equilibrium outcomes result in jurisdiction structures

[1] We will use the gender-neutral "they" and "their" for both singular and plural pronouns.

[2] We note that such economies are also called "club economies" and the local public goods are then called "club goods."

that are optimal for the economy as a whole. Of course, it also implies that economies with empty cores do not have share equilibria. Non-existence of share equilibrium for some economies is not surprising since the model is formulated in the most general way possible.

To deepen our understanding of share equilibrium and its relationship to familiar concepts, we define for each individual i their demand for public good as a function of their relative cost share in each jurisdiction to which they might belong. Note that our definition of demand is not standard because quantites demanded depend on cost shares rather than prices. When production of the public good does not exhibit constant returns to scale, cost shares lead to non-linear budget constraints. With some weak assumptions on the model, including the assumption that the local public good is not a Giffen good, we obtain the result that in a given jurisdiction, the demand for public good by a jurisdiction member i is decreasing in i's relative cost share in the jurisdiction. To our knowledge, this is a novel result and, as the reader will see, it is subtle.

We use the result that demand is decreasing in relative cost share to demonstrate that symmetric players (i.e., players who are identical) necessarily have the same cost shares in an equilibrium in which they are in the same jurisdiction and consume a positive amount of public good. Consideration of alternative jurisdictions by players leads to the result that symmetric players also have the same cost shares in an equilibrium in which they are in different multiple-member jurisdictions. These considerations allow us to provide a complete characterization of the possible variation in share indices of symmetric players that can be supported in equilibrium.

The insights on symmetric players are used to derive a share equilibrium existence result for symmetric economies, in which all individuals are identical. We formulate the property of top convexity for symmetric economies and demonstrate that this condition is both necessary and sufficient for the existence of a share equilibrium. We end the paper with a discussion of the sorts of conditions on symmetric economies that need to be satisfied in order to obtain a result that non-emptiness of the core guarantees existence of share equilibrium.

2 Local public good economies and share equilibrium

This section is devoted to formal definitions. We limit ourselves to economies with one public good and one private good.

2.1 Local public good economies

A *local public good economy*[3] is a list

$$E = \langle N; (w_i)_{i \in N}; (u_i)_{i \in N}; c \rangle,$$

where N (sometimes denoted $N(E)$) is the non-empty finite set of players in the economy, $w_i \in \mathbb{R}_+$ is the non-negative endowment of player $i \in N$ of a private good, $u_i : \mathbb{R}_+ \times \mathbb{R}_+ \times 2^N \to \mathbb{R}$ is i's utility function, and $c : \mathbb{R}_+ \times 2^N \to \mathbb{R}_+$ is the cost function for the production of local public good in jurisdictions. The family of all public good economies is denoted by $\mathcal{E}$.

In an economy E, a player i who is a member of jurisdiction $J \subseteq N$ ($i \in J$) and consumes an amount x_i of the private good and an amount y of the local public good provided in jurisdiction J, enjoys utility $u_i(x_i, y, J)$. We assume that u_i is strictly increasing in both private and (local) public good consumption.

[3] In van den Nouweland and Wooders (2011) we needed to allow for the inclusion of players that were not decision makers (so that we could use a consistency axiom). Since we will not be using consistency in the current paper, we use a slightly simplified version of the definition of economies.

In a jurisdiction J, the cost of producing y units of the local public is $c(y, J)$ units of the private good. The cost function c is non-decreasing in the level of (local) public good with $c(0, J) = 0$ for each J.

A specification of a jurisdiction structure, levels of local public good provided in each jurisdiction, and private good consumptions is called a configuration. Formally, a *configuration* in a local public good economy E with set of players N is a vector

$$(\mathbf{x}, \mathbf{y}, \mathbf{P}) = ((x_i)_{i \in N}, (y_J)_{J \in \mathbf{P}}, \mathbf{P}),$$

where $x_i \in \mathbb{R}_+$ is the consumption of the private good by player i, $\mathbf{P}$ is a partition of N into jurisdictions, and $y_J \in \mathbb{R}_+$ is the level of local public good provided in jurisdiction $J \in \mathbf{P}$. We denote the set of configurations in a local public good economy with set of players N by $C(N)$.[4] A configuration $(\mathbf{x}, \mathbf{y}, \mathbf{P}) \in C(N)$ is *feasible* if $c(y_J, J) \leq \sum_{i \in J} (w_i - x_i)$ for each $J \in \mathbf{P}$, so that the cost of public good in each jurisdiction is covered by the jurisdiction's members.

2.2 Share equilibrium

A share equilibrium (cf. van den Nouweland and Wooders, 2011) consists of a vector of share indices - one for each player in the economy - and a configuration. Share indices determine for each player i the share of the cost of the production of local public good in all potential jurisdictions that include i; if player i has share index s_i and is a member of jurisdiction $J \subseteq N$, then i pays the share $s_i / \left(\sum_{j \in J} s_j \right)$ of the cost of local public good production in jurisdiction J. Hence, share indices determine the *relative* cost shares paid by the players in each jurisdiction that might possibly be formed. A set of share indices and a configuration constitute a share equilibrium if (1) every player's membership of a jurisdiction and consumption as specified by

[4]Note that configurations do not allow for negative levels of private or public good consumption.

the configuration are utility-maximizing in their budget set as determined by their (relative) share and, moreover, (2) in every jurisdiction that is formed in an equilibrium, all members demand the same level of local public good. This implies that, in equilibrium, given the share of the cost of local public good production that they have to shoulder in various jurisdictions as determined by the share indices, each player prefers the jurisdiction to which they are assigned and the level of local public good that is provided in that jurisdiction. Moreover, in a share equilibrium, the players agree to share the cost of local public good production according to their share indices. Hence, a share equilibrium is an equilibrium in three dimensions: the cost shares arising from the players' share indices, the jurisdictions formed, and a level of local public good production for each jurisdiction that is formed. Agreement on the share indices determining cost shares, formation of jurisdictions, and levels of local public good are inextricably linked.

Formally, for a local public good economy $E = \langle N; (w_i)_{i \in N}; (u_i)_{i \in N}; c \rangle$, a set of share indices is a positive vector $\mathbf{s} = (s_i)_{i \in N} \in \mathbb{R}^N_{++}$.[5,6] For each player $i \in N$ and each jurisdiction $J \subseteq N$, player i's relative share in J is $s_i^J := s_i / \left(\sum_{j \in J} s_j \right)$. Also, if $\mathbf{P}$ is a partition of N, then for each $i \in N$ we denote the jurisdiction containing player i by $P(i)$, so that $i \in P(i) \in \mathbf{P}$.

Definition 1 *A* share equilibrium *in an economy E is a pair consisting of a vector of share indices* $\mathbf{s}$ *and a configuration* $(\mathbf{x}, \mathbf{y}, \mathbf{P})$ *such that for each*

[5] Assuming share indices to be positive is without loss of generality. It follows from condition 2 of share equilibrium and players' utility functions being strictly increasing in local public good consumption that there can be no share equilibrium in which a player has a share index equal to zero as this would imply that a player always wants to consume more of the local public good, which is free to them. For similar reasons, it is impossible to have some players who are subsidized, i.e. have a negative share index.

[6] We could normalize the share indices to sum to 1. However, this is unnecessary for any of our results and would complicate notation in the paper. Moreover, such a normalization is inherently arbitrary - for example, we may as well normalize the share indices such that they sum up to the number of players. Therefore, we have opted not to normalize the share indices.

$i \in N(E)$ the following two conditions are satisfied[7]

1. $s_i^{P(i)} c(y_{P(i)}, P(i)) + x_i = w_i$
2. *for all* $(\overline{x}_i, \overline{y}, \overline{J}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times 2^N$ *such that* $i \in \overline{J}$ *and* $s_i^{\overline{J}} c(\overline{y}, \overline{J}) + \overline{x}_i \leq w_i$*, it holds that* $u_i(x_i, y_{P(i)}, P(i)) \geq u_i(\overline{x}_i, \overline{y}, \overline{J})$.

If $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))$ is a share equilibrium, then the components of $\mathbf{s}$ are *equilibrium share indices* and $(\mathbf{x}, \mathbf{y}, \mathbf{P})$ is an *equilibrium configuration*. The set of share equilibria of an economy E is denoted $SE(E)$. Note that the share indices appear only in a relative manner, so that if $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))$ is a share equilibrium in an economy E, and $\alpha > 0$, then $(\alpha\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))$ is also a share equilibrium in economy E. In this sense, a share equilibrium is never unique.

3 Share equilibrium and the core

In this section we explore relations between share equilibria and the core of a local public good economy and we prove core inclusion of share equilibrium configurations.

The core of an economy is the set of configurations that are stable against deviations by coalitions players. When a coalition deviates, its members can form new jurisdictions and within each of these jurisdictions the members can decide on a level of local public good to be provided and on a way to share the cost of its provision among the jurisdiction members. Because there are no externalities between jurisdictions, it obviously is not optimal for the members of a jurisdiction to subsidize the cost of the local public

[7] The equality in condition 1 (the budget constraint) could be a weak inequality, but using equality is without loss of generality because a player who is not spending all of their available endowment would always be able to consume more of the private good and obtain a higher utility. Therefore condition 2 (the utility-maximization constraint) cannot be satisfied if $s_i^{P(i)} c(y_{P(i)}, P(i)) + x_i < w_i$.

good in another jurisdiction, and thus all jurisdictions will have to be self-sufficient. This implies that we can limit ourselves to considering deviations in which a deviating coalition forms one new jurisdiction that includes all members of the coalition, because a configuration is stable against such deviations if and only if it is stable against deviations by coalitions that can form multiple new jurisdictions amongst themselves.

Definition 2 *The* core *of an economy* $E = \langle N; (w_i)_{i\in N}; (u_i)_{i\in N}; c\rangle$ *is the set of all configurations* $(\mathbf{x}, \mathbf{y}, \mathbf{P}) \in C(N)$ *such that for every* $T \subseteq N$*, private-good consumption levels* $(\overline{x}_i)_{i\in T} \in \mathbb{R}^T_+$*, and local public good level* $\overline{y} \in \mathbb{R}_+$ *satisfying* $c(\overline{y}, T) \leq \sum_{i\in T}(w_i - \overline{x}_i)$*, it holds that, if there exists a player* $i \in T$ *who is strictly better off (that is, a player* i *for whom* $u_i(x_i, y_{P(i)}, P(i)) < u_i(\overline{x}_i, \overline{y}, T)$*), then there exists a player* $j \in T$ *who is strictly worse off (that is, a player* j *for whom* $u_j(x_j, y_{P(j)}, P(j)) > u_j(\overline{x}_j, \overline{y}, T)$*).*

Theorem 1 shows that every equilibrium configuration is in the core. Results in the same spirit hold in other contexts.[8] However, our model allows consumption and production possibilities, as well as preferences, to depend on the specific identities of individuals jointly producing and consuming the public good in each jurisdiction. This creates new possibilities for deviations. For example, a rich person with an unlikable personality could try to entice a poor person who is very desirable as a jurisdiction member on the basis of personal characteristics, to enter into a jurisdiction by offering to pay for large quantities of public good and to give the poor person additional private good quantities. As long as the outlays of the rich person do not exceed their endowments, this is an allowed deviation under core considerations. In

[8] To name a few: Debreu and Scarf (1963, Theorem 1) shows that price-taking equilibrium outcomes are in the core of a private-goods exchange economy; Foley (1970) shows that Lindahl equilibrium outcomes are in the core of an economy with public and private goods; Kaneko (1977) notes that ratio equilibrium outcomes are in the core of a voting game in a public goods economy; Wooders (1978) and, most recently, Allouch and Wooders (2008) demonstrate that price-taking equilibrium outcomes are in the core of a club economy.

a share equilibrium, however, the poor player may have a small share index and enjoy a large amount of the public good, but they cannot receive a transfer of private good. Core inclusion of share equilibrium configurations is thus not an obvious result.

Theorem 1 *Let* $E = \langle N; (w_i)_{i\in N}; (u_i)_{i\in N}; c\rangle \in \mathcal{E}$ *be a local public good economy, and let the share indices* $(s_i)_{i\in N}$ *and configuration* $((x_i)_{i\in N}, (y_J)_{J\in\mathbf{P}}, \mathbf{P})$ *constitute a share equilibrium in* E*. Then the configuration* $(\mathbf{x}, \mathbf{y}, \mathbf{P})$ *is in the core of* E*.*

Proof. Take a coalition $T \subseteq N$ and an allocation of private and local public good consumption $((\overline{x}_i)_{i\in T}, \overline{y})$ satisfying

$$c(\overline{y}, T) \leq \sum_{i\in T} (w_i - \overline{x}_i)\,. \tag{1}$$

Suppose that for player $i \in T$ it holds that $u_i(x_i, y_{P(i)}, P(i)) < u_i(\overline{x}_i, \overline{y}, T)$. Then because $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \in SE(E)$, we know by condition 2 of share equilibrium that

$$s_i^T c(\overline{y}, T) + \overline{x}_i > w_i. \tag{2}$$

On other hand,

$$\sum_{j\in T} \left(s_j^T c(\overline{y}, T) + \overline{x}_j\right) = c(\overline{y}, T) + \sum_{j\in T} \overline{x}_j \leq \sum_{j\in T} w_j, \tag{3}$$

where the inequality follows from affordability assumption (1). Putting (3) together with (2), we find that for some player $j \in T$ it is the case that

$$s_j^T c(\overline{y}, T) + \overline{x}_j < w_j. \tag{4}$$

Thus, in jurisdiction T player j can afford to consume $\overline{y}$ and

$$\hat{x}_j := w_j - s_j^T c(\overline{y}, T) > \overline{x}_j.$$

It follows from condition 2 of the share equilibrium $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))$ and the assumption that a player's utility function is strictly increasing in private good consumption that

$$u_j(x_j, y_{P(j)}, P(j)) \geq u_j(\hat{x}_j, \overline{y}, T) > u_j(\overline{x}_j, \overline{y}, T).$$

Hence, j would be strictly worse off by belonging to T with the proposed allocation. ■

Theorem 1 contributes two elements to our understanding of share equilibrium. On the one hand share equilibria are desirable because they select core allocations. On the other hand share equilibria will not exist in economies that have empty cores. The theorem does not contribute to our understanding of the potential variation in share equilibria. We consider this topic in the following sections.

4 Share indices and demand for public good

In the current section, we investigate the effect of share indices on demand for public good. This is important to understand because in a share equilibrium the levels of local public good production for the jurisdictions that are formed have to be optimal given the cost shares arising from the players' equilibrium share indices.

Definition 3 *Let E be an economy, let i be a player in E, and let $J \subseteq N$ be a jurisdiction with $i \in J$. Let* $\mathbf{s}$ *be a vector of share indices. Then i's* demand for public good *in jurisdiction J as a function of i's relative cost share s_i^J is*

$$D_i^J(s_i^J) := argmax_{y \in \mathbb{R}_+ :\, s_i^J c(y,J) \leq w_i} u_i(w_i - s_i^J c(y,J), y, J).$$

Note that in certain cases there may be more than one utility-maximizing level of public good. Rather than defining the demand as a correspondence, in such cases we choose to simply pick one of the possible levels of public good. This is not going to influence our results, but it does cut down on technicalities that are necessary in proofs. Note that $D_i^J(\cdot)$ is different from a demand function in the usual sense because the relative cost share s_i^J is not a price and it does not give rise to a linear budget constraint (unless, of course, in cases where the cost function itself is linear). However, under some common and mild constraints, we can use our knowledge of price-based demand to make inferences about demand as a function of cost shares.

Assumption 1 *There are three parts to this assumption, regarding utility functions, cost functions, and price-based demand.*

- ***a.*** *For each player $i \in N$ and each jurisdiction $J \subseteq N$ with $i \in J$, it holds that the utility function $u_i(\cdot, \cdot, J)$ is strictly increasing, differentiable, and quasi concave (as a function of x_i and y, keeping the jurisdiction fixed).*

- ***b.*** *For each jurisdiction $J \subseteq N$, it holds that the cost function $c(\cdot, J)$ is strictly increasing, twice differentiable, and convex (as a function of y, keeping the jurisdiction fixed), and $c(0, J) = 0$.*

- ***c.*** *The local public good y is not a Giffen good.*

Note that the basic assumptions on local public good economies (utility functions are strictly increasing in both private and public good consumption, the cost function is non-decreasing in the level of public good, and $c(0, J) = 0$ for each J) are subsumed in Assumption 1. Parts a and b of Assumption 1 are standard throughout the literature. They amount to assuming that preferences over private and public good consumption are convex, and that costs for producing public good are increasing at a non-decreasing rate. Part c of Assumption 1 can be obtained as a result of various more basic assumptions on the utility functions (for example, the public good is not inferior, or the utility functions $u_i(\cdot, \cdot, J)$ are homothetic), but the current formulation is the weakest we can find that allows us to derive that demands for public good are decreasing in cost shares.

Assumption 1 and the proof of the following theorem are carefully designed to allow for a linear cost function as well as a utility function that gives rise to indifference curves that have linear sections. This means that

Figure 1: y a Giffen good

our results are applicable to the extreme cases where the production of public good exhibits constant returns to scale and/or the private good and the public good are perfect substitutes for some player.

Theorem 2 ***Downward sloping demand*** *Let E be an economy that satisfies Assumption 1, let $i \in N$ be a player in E, and let $J \subseteq N$ be a jurisdiction with $i \in J$. Then $D_i^J(\cdot)$ is decreasing in i's relative cost share, and strictly decreasing whenever i demands a positive amount of public good.*

Proof. Because J is fixed throughout this proof, we will omit it as an index on cost shares and i's demand, and as an argument in the cost function. We will be considering player i's consumption of private and public good, and this can be visualized in a 2-dimensional axes system with y on the horizontal axis and x_i on the vertical axis, as in Figure 1. The reader can refer to the figure for a graphical exposition of this proof.

For any relative cost share $r_i \in (0,1]$ for i in J, define the *budget constraint* $BC(r_i)$ by

$$BC(r_i) := \{(y, x_i) \in \mathbb{R}^2_+ \mid x_i = w_i - r_i c(y)\}.$$

Clearly, $BC(r_i)$ intercepts the x_i-axis at w_i (because $c(0) = 0$) and the y-axis at $y = c^{-1}(\frac{w_i}{r_i})$. Because $c(\cdot)$ is strictly increasing, twice differentiable, and convex, $BC(r_i)$ is strictly decreasing, twice differentiable, and concave.[9]

Let $r_i \in (0,1]$ be a relative cost share of player i and $\tilde{y}$ a quantity of public good such that $r_i c(\tilde{y}) \leq w_i$. The line $L(r_i, \tilde{y})$ through $(\tilde{y}, w_i -$

[9] Strictly speaking, we should state that $BC(r_i)$ implicitly defines x_i as a function of y and it is this function that is strictly decreasing, twice differentiable, and concave. In order to keep the exposition of an already intricate proof as simple as possible, we will not make such subtle distinctions explicit.

$r_i c(\tilde{y})$) and tangent to $BC(r_i)$ has slope $-r_i c'(\tilde{y})$ and intercepts the x_i-axis at $I(r_i, \tilde{y}) = w_i - r_i c(\tilde{y}) + r_i c'(\tilde{y})\tilde{y}$. Due to the concavity of $BC(r_i)$, it holds that $I(r_i, \tilde{y}) \geq w_i$. Moreover, $I(r_i, \tilde{y}) = w_i$ only if $\tilde{y} = 0$ or if c is linear on $[0, \tilde{y}]$, and in all other cases the x_i-intercept $I(r_i, \tilde{y})$ is outside the budget set $(I(r_i, \tilde{y}) > w_i)$.

Note that it follows from Assumption 1.a that i's indifference curves are strictly downward sloping and convex. Thus, player i's demand for public good $D_i(r_i)$ can graphically be found as the y-coordinate of a point where i's indifference curve is tangent to the budget constraint $BC(r_i)$ if such a point exists, and as a corner solution otherwise.[10] Because the player is fixed throughout this proof, we will simplify notation and write $y(r_i)$ instead of $D_i(r_i)$.

Define $IC(r_i)$ to be the indifference curve through $(y(r_i), w_i - r_i c(y(r_i)))$, and $BL(r_i)$ to be the tangent line $L(r_i, y(r_i))$. Defining $P(r_i)$ to be equal to $r_i c'(y(r_i))$, we obtain that $BL(r_i)$ is described by $x_i = I(r_i, y(r_i)) - P(r_i)y$.[11] By construction, if player i had an endowment $I(r_i, y(r_i))$ of the private good, and could purchase the public good at a market price of $P(r_i)$ per unit, then player i's budget line would equal $BL(r_i)$ and $(y(r_i), w_i - r_i c(y(r_i)))$ would be a utility-maximizing consumption bundle for player i.

Now we are set up to use our knowledge of familiar price-based models to make inferences about the shares-based demand function $D_i(\cdot)$. Consider two different relative cost shares $\overline{r}_i$ and $\hat{r}_i$ for player i with the property that $0 < \overline{r}_i < \hat{r}_i \leq 1$. Suppose that $y(\hat{r}_i) \geq y(\overline{r}_i) > 0$, i.e., that i's demand for public good is *not* strictly decreasing with i's relative cost share.[12] We will

[10] The possibility of corner solutions, i.e., where $D_i(r_i) = 0$ or where $D_i(r_i) = c^{-1}(\frac{w_i}{r_i})$, is important to consider. We discuss these separately only when necessary.

[11] Note that if there are multiple utility-maximizing levels of public good at the relative cost share r_i, then both $IC(r_i)$ and $BC(r_i)$ must be linear for all $y \in D_i(r_i)$, so that all possible choices of $y(r_i)$ result in the same line $BL(r_i)$.

[12] We need to separately cover the possibility that $y(\overline{r}_i) = 0$, in which case $D_i(\cdot)$ cannot be required to be *strictly* decreasing. We point out in the next footnote what adjustments need to be made to demonstrate that in this case $y(\hat{r}_i) = 0$ has to hold.

demonstrate that this leads to a contradiction with Assumption 1.c.

If $c(\cdot)$ is linear on the segment $[0, y(\hat{r}_i)]$, then both $BC(\overline{r}_i)$ and $BC(\hat{r}_i)$ are linear when restricted to the region where $y \in [0, y(\hat{r}_i)]$ and then $\hat{r}_i > \overline{r}_i > 0$ and $y(\hat{r}_i) \geq y(\overline{r}_i) > 0$ contradict the assumption that the public good is not a Giffen good.

Suppose that $c(\cdot)$ is not linear on the segment $[0, y(\hat{r}_i)]$. We consider the two derived budget lines $BL(\overline{r}_i)$ and $BL(\hat{r}_i)$. Because $c(0) = 0$ and $c(\cdot)$ is strictly increasing and convex, it follows for all $\tilde{y} > 0$ and $r_i > 0$ that $\frac{\partial I(r_i,\tilde{y})}{\partial r_i} = c'(\tilde{y})\tilde{y} - c(\tilde{y}) \geq 0$, where this can only hold with equality if $c(\cdot)$ is linear on the segment $[0, \tilde{y}]$, and $\frac{\partial I(r_i,\tilde{y})}{\partial \tilde{y}} = r_i c''(\tilde{y})\tilde{y} \geq 0$. Combining this with $\hat{r}_i > \overline{r}_i > 0$ and $y(\hat{r}_i) \geq y(\overline{r}_i) > 0$, as well as the assumption that $c(\cdot)$ is not linear on the segment $[0, y(\hat{r}_i)]$, it follows that $I(\hat{r}_i, y(\hat{r}_i)) > I(\overline{r}_i, y(\overline{r}_i))$.[13]

Define $L(\overline{r}_i, \hat{r}_i)$ be the line through $(0, I(\overline{r}_i, y(\overline{r}_i)))$ on $BL(\overline{r}_i)$ and the point $(y(\hat{r}_i), w_i - \hat{r}_i c(y(\hat{r}_i)))$ on $BC(\hat{r}_i)$. By construction, the line $L(\overline{r}_i, \hat{r}_i)$ is flatter than the tangent line $BL(\hat{r}_i)$[14] and steeper than $BL(\overline{r}_i)$[15], and thus has equation $x_i = I(\overline{r}_i, y(\overline{r}_i)) - Py$ with $P(\overline{r}_i) < P < P(\hat{r}_i)$.

The line $L(\overline{r}_i, \hat{r}_i)$ would be player i's budget line if the player had an endowment $I(\overline{r}_i, y(\overline{r}_i))$ of the private good, and could purchase the public good at a market price of P per unit. Under these conditions, player i would demand an amount $y > y(\hat{r}_i)$ of the public good, because at the point $(y(\hat{r}_i), w_i - \hat{r}_i c(y(\hat{r}_i)))$ the line $L(\overline{r}_i, \hat{r}_i)$ crosses player i's indifference curve $IC(\hat{r}_i)$ and is flatter than that indifference curve.

Summarizing, if player i has an endowment $I(\overline{r}_i, y(\overline{r}_i))$ of the public good and can purchase the public good at the market price of $P(\overline{r}_i)$ per unit, then i's utility-maximizing consumption of the public good is $y(\overline{r}_i)$. However, if player i has an endowment $I(\overline{r}_i, y(\overline{r}_i))$ of the public good and can purchase

[13] If $y(\overline{r}_i) = 0$ and $y(\hat{r}_i) > y(\overline{r}_i)$, then $I(\hat{r}_i, y(\hat{r}_i)) > w_i = I(\overline{r}_i, y(\overline{r}_i))$. The continuation of the proof covers this extreme case.

[14] This follows immediately from $I(\hat{r}_i, y(\hat{r}_i)) > I(\overline{r}_i, y(\overline{r}_i))$.

[15] Note that $0 < \overline{r}_i < \hat{r}_i$ and $y(\hat{r}_i) > 0$ imply that the point $(y(\hat{r}_i), w_i - \hat{r}_i c(y(\hat{r}_i)))$ on budget constraint $BC(\hat{r}_i)$, is strictly below the budget constraint $BC(\overline{r}_i)$ and thus also strictly below the tangent line $BL(\overline{r}_i)$.

the public good at the higher market price of P per unit, then i's utility-maximizing consumption of the public good is some $y > y(\hat{r}_i) \geq y(\overline{r}_i)$. This contradicts the assumption that the public good is not a Giffen good. ■

Within a jurisdiction, a careful balance of the relative share indices of the members is necessary in order for the jurisdiction members to be able to reach agreement on the level of public good to be provided in the jurisdiction. Our downward-sloping demand result Theorem 2 implies that to reach such a balance, the share indices of players who want more public good need to be raised relative to the share indices of players who want less public good. However, if the relative share of one of the players becomes too high, then such a player will want to withdraw from the jurisdiction and form an alternative jurisdiction with players who are willing to pay more for public good. It is these various pressures that shape equilibrium share indices.

5 Symmetric players

In order to further develop our intuition regarding share equilibrium, we consider possible variation in share indices of symmetric players. Symmetric players are different only in name, and have indistinguishable influences on costs and utilities. In the current section, we investigate to what extent such players can have different share indices in equilibrium.

Definition 4 *Two players i and j are* symmetric *in local public good economy $E = \langle N; (w_i)_{i\in N}; (u_i)_{i\in N}; c\rangle \in \mathcal{E}$ if*

1. $w_i = w_j$
2. *for all $y \in \mathbb{R}_+$ and all $J \subseteq N\backslash\{i, j\}$, it holds that $c(y, J\cup i) = c(y, J\cup j)$*
3. *for all $x \in \mathbb{R}_+$, $y \in \mathbb{R}_+$, and $J \subseteq N\backslash\{i, j\}$:*

$$u_i(x, y, J \cup i) = u_j(x, y, J \cup j)$$

and for all $x \in \mathbb{R}_+$, $y \in \mathbb{R}_+$, *and* $J \subseteq N$ *with* $\{i,j\} \subseteq J$*:*

$$u_i(x,y,J) = u_j(x,y,J)$$

4. *for all* $k \in N \backslash \{i,j\}$, $x \in \mathbb{R}_+$, $y \in \mathbb{R}_+$, *and* $J \subseteq N \backslash \{i,j\}$*:*

$$u_k(x,y,J \cup i) = u_k(x,y,J \cup j)$$

In the following series of results, we explore circumstances under which symmetric players necessarily have the same share index in a share equilibrium. The first result in this sequence shows that if there are two symmetric players i and j in different jurisdictions and at least one of these players has a fellow jurisdiction member k, then the share indices of the two symmetric players have to be the same.[16] The intuition behind the result is different depending on two circumstances. In a share equilibrium, no player $k \neq i,j$ wants to be in a jurisdiction with player i if i has a lower share index than player j, because player j would pay a larger share of the cost of local public good production, while playing the same role as player i in other players' utility functions as well as in the cost of public good production. If player i has a lower share index than player j and i is already in a jurisdiction by themselves, then player i pays all of the cost of local public good production in their jurisdiction. Thus, player j can mimic player i, but has no incentive to do so in a share equilibrium. On the other hand, player i could do better than player j by replacing them in their jurisdiction, but with a lower share of the cost of local public good production than player j.

Theorem 3 *Let* E *be an economy and* i *and* j *symmetric players in* E*. Let* $(\mathbf{s},(\mathbf{x},\mathbf{y},\mathbf{P}))$ *be a share equilibrium in economy* E *with* $i \notin P(j)$ *and* $\exists k \in N \backslash \{i,j\}$ *with* $k \in P(j)$*. Then* $s_i = s_j$*.*

[16] Prior theorems in the literature on games and local public good economies may appear to be quite closely related; see, for example, Wooders (1983), Theorem 3, but there is a significant difference. The prior theorems rely on the assumption that all gains to collective activities can be realized by groups bounded in size and these are smaller than the entire economy. No such restriction is made here.

Proof. We will rule out the possibilities that $s_i > s_j$ or $s_j > s_i$.

First, suppose $s_i > s_j$. Let $k \in P(j)$, $k \neq j$. We will show that player k can do better in a jurisdiction that replaces player j with player i. We derive the following string of (in)equalities.

$$\begin{aligned} w_k &= s_k^{P(j)}\, c(y_{P(j)}, P(j)) + x_k \\ &= \frac{s_k}{\sum_{l \in P(j) \backslash j} s_l + s_j}\, c(y_{P(j)}, P(j)) + x_k \\ &> \frac{s_k}{\sum_{l \in P(j) \backslash j} s_l + s_i}\, c(y_{P(j)}, P(j)) + x_k \\ &= s_k^{(P(j)\backslash j) \cup i}\, c(y_{P(j)}, (P(j) \backslash j) \cup i) + x_k, \end{aligned}$$

where the first equality follows from condition 1 of the share equilibrium, the second equality follows from the definition of the relative shares of players in jurisdictions, the inequality follows because $s_i > s_j$ by assumption, and the third equality follows from the definition of the relative shares of players in jurisdictions and condition 2 of symmetry. It follows that player k can afford to consume more than the quantity x_k of the private good in jurisdiction $\overline{J} := (P(j) \backslash j) \cup i$, namely $\overline{x}_k := w_k - s_k^{(P(j)\backslash j)\cup i}\, c(y_{P(j)}, (P(j) \backslash j) \cup i)$. Because u_k is strictly increasing in private good consumption by player k, we know

$$u_k\left(\overline{x}_k, y_{P(j)}, \overline{J}\right) > u_k(x_k, y_{P(j)}, \overline{J}). \tag{5}$$

By condition 4 of symmetry, it follows that

$$u_k(x_k, y_{P(j)}, \overline{J}) = u_k(x_k, y_{P(j)}, P(j)). \tag{6}$$

Hence, we have $s_k^{\overline{J}}\, c(y_{P(j)}) + \overline{x}_k = w_k$ and $u_k\left(\overline{x}_k, y_{P(j)}, \overline{J}\right) > u_k(x_k, y_{P(j)}, P(j))$. This violates condition 2 of the share equilibrium $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))$. Thus, we see that the assumption $s_i > s_j$ leads to a contradiction.

Now, suppose $s_j > s_i$. First, note that it follows from what we have shown so far, that we can assume without loss of generality that $P(i) =$

$\{i\}$.[17] Note that player j can afford to mimic player i, because

$$\begin{aligned} s_j^{\{j\}}\, c(y_{\{i\}}, \{j\}) + x_i &= c(y_{\{i\}}, \{j\}) + x_i \\ &= c(y_{\{i\}}, \{i\}) + x_i \\ &= s_i^{\{i\}}\, c(y_{\{i\}}, \{i\}) + x_i \\ &= w_i = w_j, \end{aligned}$$

where the first and third equalities follow from the definition of the relative shares of players in jurisdictions, the second and fifth equalities follow from symmetry of players i and j, and the fourth equality follows from condition 1 of the share equilibrium. It follows from condition 2 of the share equilibrium that player j is no better off when they mimic player i. Using this and also symmetry of players i and j, we derive

$$u_j\left(x_j, y_{P(j)}, P(j)\right) \geq u_j\left(x_i, y_{P(i)}, \{j\}\right) = u_i\left(x_i, y_{\{i\}}, \{i\}\right). \tag{7}$$

On the other hand, player i can improve their utility by taking player j's position. To see this, first note that

$$\begin{aligned} s_i^{(P(j)\backslash j)\cup i}\, c\left(y_{P(j)}, (P(j)\backslash j) \cup i\right) + x_j &= \frac{s_i}{\sum_{l\in P(j)\backslash j} s_l + s_i}\, c(y_{P(j)}, P(j)) + x_j \\ &< s_j^{P(j)}\, c\left(y_{P(j)}, P(j)\right) + x_j \\ &= w_j = w_i, \end{aligned}$$

where the first and third equalities follow from symmetry of players i and j, the strict[18] inequality follows from the assumption $s_i < s_j$, and the second equality follows from condition 1 of the share equilibrium. It follows that player i can afford to be in jurisdiction $\overline{J} := (P(j)\backslash j) \cup i$ consuming $y_{P(j)}$ of the local public good, while consuming more than the quantity x_j of the private good, namely $\overline{x}_i := w_i - s_i^{\overline{J}}\, c(y_{P(j)}, \overline{J}) > x_j$. Hence, it follows from

[17] Otherwise, we can consider a member of $P(i)\backslash i$ and proceed as before.

[18] Note that $k \in P(j)\backslash j$, so that $\sum_{l\in P(j)\backslash j} s_l > 0$.

condition 2 of the share equilibrium that

$$\begin{aligned} u_i(x_i, y_{\{i\}}, \{i\}) &\geq u_i\left(\overline{x}_i, y_{P(j)}, \overline{J}\right) \\ &= u_j\left(\overline{x}_i, y_{P(j)}, P(j)\right) \\ &> u_j(x_j, y_{P(j)}, P(j)), \end{aligned} \tag{8}$$

where the equality follows from the symmetry of players i and j, and the strict inequality follows because u_j is strictly increasing in private good consumption by player j. Clearly, (7) and (8) cannot both be true. Thus, the assumption $s_j > s_i$ also leads to a contradiction. ■

In the previous theorem, the equilibrating factor that forces the share indices of two symmetric players to be the same in a share equilibrium is that players consider alternative jurisdictions. If the two symmetric players are in the same jurisdiction, these considerations do not have any bite. However, there is another equilibrating factor at work in this case, which is that the two symmetric players each have to be consuming the same amount of local public good. Using the downward sloping demand theorem, we can show that the players' consideration of alternative amounts of local public good forces the share indices of two symmetric players in the same jurisdiction to be the same if a positive amount of the public good is produced in the jurisdiction.

Theorem 4 *Let E be an economy that satisfies Assumption 1 and let i and j be symmetric players in E. Let $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))$ be a share equilibrium in economy E with $P(i) = P(j)$ and $y_{P(i)} > 0$. Then $s_i = s_j$.*

Proof. Condition 2 of the share equilibrium applied to player i and jurisdiction $P(i)$ implies that $(x_i, y_{P(i)})$ is a solution to the optimization problem

$$\begin{array}{ll} \text{maximize} & u_i\left(\overline{x}, \overline{y}, P(i)\right) \\ \text{subject to} & \overline{x} \in \mathbb{R}_+ \\ & \overline{y} \in \mathbb{R}_+ \\ & s_i^{P(i)} c\left(\overline{y}, P(i)\right) + \overline{x} \leq w_i \end{array} \tag{9}$$

Or, stated differently, $y_{P(i)} = D_i^{P(i)}(s_i^{P(i)})$ as defined in Definition 3.

Because $P(i) = P(j)$, and thus $\{i, j\} \subseteq P(i)$, we can use symmetry of i and j to derive that $u_i(\overline{x}, \overline{y}, P(i)) = u_j(\overline{x}, \overline{y}, P(i))$ for all $\overline{x} \in \mathbb{R}_+$ and $\overline{y} \in \mathbb{R}_+$. In addition, we know that $w_i = w_j$. Substituting j for i in the appropriate places in (9), we derive that $D_j^{P(i)}(r) = D_i^{P(i)}(r)$ for all $r \in (0,1)$. Hence, if the symmetric players i and j have the same relative cost shares in jurisdiction $P(i)$, then they demand the same amount of public good in that jurisdiction.[19]

By assumption $y_{P(i)} > 0$. Theorem 2 thus tells us that $D_i^{P(i)}$ is decreasing in i's relative cost share and that it is strictly decreasing at $s_i^{P(i)}$. Combining this with $D_j^{P(i)}(r) = D_i^{P(i)}(r)$ for all $r \in (0,1)$, we derive that for j to demand the level $y_{P(i)}$ of public good in the jurisdiction $P(i)$, it has to hold that $s_j^{P(i)} = s_i^{P(i)}$.

Hence, $1 = \frac{s_j^{P(i)}}{s_i^{P(i)}} = \frac{s_j}{s_i}$ and thus $s_i = s_j$. ■

Note that we use Theorem 2 to argue that it is impossible to have two symmetric players with different share indices who demand the same amount of local public good in a jurisdiction that includes them both. Assumption 1 is invoked in Theorem 4 in order to be able to apply the downward sloping demand theorem. It is possible to replace Assumption 1 in the statement of Theorem 4 by any other assumptions that guarantee that a player who has to pay more for local public good production will demand less of the good.

We have shown that symmetric players necessarily have the same share index in a share equilibrium in which they are in different jurisdictions and at least one of them has a fellow jurisdiction member and, under some common assumptions, also in a share equilibrium in which they are in the same jurisdiction and a positive amount of local public good is produced in that jurisdiction. In the following two examples, we demonstrate that symmetric

[19] Note that $D_j^{P(i)}(r) = D_i^{P(i)}(r)$ holds as an equality of sets if the utility-maximizing level of public good is not unique. This poses no problem in the continuation of the proof.

players may have different share indices if they either form singleton jurisdictions (providing their own local public good, which then essentially becomes a private good) or they end up in the same jurisdiction and no local public good is provided in this jurisdiction.

Example 1 *Consider a local public good economy with two symmetric players who have utility functions such that, all else equal, they prefer to be alone. We demonstrate that share equilibria exist in which the two players each form a singleton jurisdiction and have different share indices.*

Let $E = \langle N; (w_i)_{i\in N}; (u_i)_{i\in N}; c\rangle$ *be the local public good economy with* $N = \{1, 2\}$, $w_1 = w_2 = 2$, $u_i(x_i, y, J) = x_i^{1/2} y^{1/2} - (|J| - 1)$, *and* $c(y, J) = |J|\, y$. *Note that economy* E *satisfies Assumption 1.*

For a jurisdiction consisting of one player, the utility-maximizing level of local public good is easily seen to be $y = 1$, *so that the single player in the jurisdiction consumes equal amounts of private and local public good and has utility* $u_i(1, 1, \{i\}) = 1 - (1 - 1) = 1$.

For the 2-player jurisdiction $\{1, 2\}$, *the utility-maximizing levels of consumption are found as the solution to*

$$\begin{array}{ll} \textit{maximize} & x_1^{1/2} y^{1/2} + x_2^{1/2} y^{1/2} - 2\,(2 - 1) \\ \textit{subject to} & x_1 + x_2 + 2y = 4 \end{array}$$

The solution is $x_1 = x_2 = 1$ *and* $y = 1$, *resulting in the maximum utility of* $u_1(1, 1, \{1, 2\}) + u_2(1, 1, \{1, 2\}) = 2(1 - (2 - 1)) = 0$.

Because share equilibrium configurations are in the core of the economy (per Theorem 1), we find that all share equilibria have two jurisdictions, namely $\{1\}$ *and* $\{2\}$, *and private and local public good levels* $x_1^* = x_2^* = 1$ *and* $y^*_{\{1\}} = y^*_{\{2\}} = 1$.

The players do not have to have equal share indices to get this configuration. To understand this, first note that in a singleton jurisdiction, the single member always pays for the full cost of local public good production, *regardless of the share indices (because* $s_i^{\{i\}} = 1$*). Hence, for the share equilibrium configuration, condition 1 of the share equilibrium is satisfied for any share indices. Hence, any share indices that are consistent with condition 2 of the share equilibrium for both players will give us a share equilibrium. Thus, we only need to make sure that the share indices are close enough so that the player* i *with the lower share index does not have to pay so low a share of local public good production in a jurisdiction that includes the other player,* j*, that* i *has an incentive to join* j*.*

For example, the share indices $s_1 = 1$ *and* $s_2 = 1/2$ *work. With these share indices, player* 2 *has to pay* $1/3$ *of the cost of local public good production in jurisdiction* $\{1, 2\}$*. Player 2's demand for private and local public good in jurisdiction* $\{1, 2\}$ *is then found as the solution to*

$$\begin{array}{ll} \textit{maximize} & x_2^{1/2} y^{1/2} \\ \textit{subject to} & \frac{1}{3}(2y) + x_2 = 2 \end{array}$$

The solution to this maximization problem is $\overline{y} = \frac{3}{2}$ *and* $\overline{x}_2 = 1$*, which results in a maximum utility of* $u_2\left(1, \frac{3}{2}, \{1, 2\}\right) = \sqrt{3/2} - 1 < 1$*. Thus, player 2 has a higher utility in jurisdiction* $\{2\}$*. It follows straightforwardly that player 1, who has to pay a larger share of the cost of local public good production in jurisdiction* $\{1, 2\}$ *than player 2 has to pay, also has a higher utility in a jurisdiction by themselves than in a jurisdiction that they share with player 2.*

So, the share indices $s_1 = 1$ *and* $s_2 = 1/2$ *work. Of course, equal shares work too. In addition, fixing* $s_1 = 1$*, it follows immediately from what we have shown above that 2's share index can be anything between 1/2 and 1 and, moreover, that the bound 1/2 is not sharp. Also, because of symmetry between the two players, we know that all that matters is the* ratio *between the two share indices, so that we can conclude that with* $s_1 = 1$ *any* s_2 *between 1 and 2 also works.*

In the following example we demonstrate that symmetric players may have different share indices if they end up in the same jurisdiction and no local public good is provided in this jurisdiction.

Example 2 *Consider a local public good economy with two symmetric players who have utility functions such that, all else equal, they prefer to be in a jurisdiction with others. Also, the costs of local public good production are so high that the players do not find it worth their while to produce a positive amount.*

Let $E = \langle N; (w_i)_{i\in N}; (u_i)_{i\in N}; c\rangle$ *be the local public good economy with* $N = \{1,2\}$, $w_1 = w_2 = 2$, $u_i(x_i, y, J) = x_i + y + (|J| - 1)$, *and* $c(y, J) = 2\,|J|\,y$. *Economy* E *satisfies Assumption 1.*

For a jurisdiction consisting of one player, the utility-maximizing levels of consumption of private and local public good are found as the solution to

$$\begin{array}{ll} \textit{maximize} & x_i + y + (1-1) \\ \textit{subject to} & x_i + 2y = 2 \\ & x_i \geq 0,\ y \geq 0 \end{array}$$

The solution is $x_1 = 2$ *and* $y = 0$, *resulting in the maximum utility of* $u_i(2, 0, \{1\}) = 2 + 0 + (1-1)) = 2$.

For the 2-player jurisdiction $\{1,2\}$, *the utility-maximizing levels of consumption are found as the solution to*

$$\begin{array}{ll} \textit{maximize} & x_1 + y + x_2 + y + 2(2-1) \\ \textit{subject to} & x_1 + x_2 + 4y = 4 \\ & x_1,\ x_2 \geq 0,\ y \geq 0 \end{array}$$

The solutions satisfy $y = 0$ *and* $x_1 + x_2 = 4$, *resulting in the maximum utility of* $4 + 0 + 2(2-1) = 6$.

Because share equilibrium configurations are in the core of the economy (per Theorem 1), we find that all share equilibria have one jurisdiction, namely $\{1,2\}$, *local public good level* $y^*_{\{1,2\}} = 0$, *and private good levels* $x_1^* = x_2^* = 2$.

The players do not have to have equal share indices to get this configuration. First note that the cost of producing no public good equals $c(0, \{1, 2\}) = 0$*, so that condition 1 of the share equilibrium is satisfied for any share indices. Also, each player has utility* $u_1(2, 0, \{1, 2\}) = u_2(2, 0, \{1, 2\}) = 3$*.*

Regardless of the share indices, a player in a singleton jurisdiction always pays for the full cost of local public good production, and we have already seen that this leads to a maximum utility of 2. Hence, regardless of their share indices, the players are better off in the configuration $(x^*, y^*, \{1, 2\})$*.*

We conclude that any share indices that guarantee that no player wants to see a different level of local public good being produced in jurisdiction $\{1, 2\}$*, are consistent with condition 2 of the share equilibrium.*

For example, the share indices $s_1 = 1$ *and* $s_2 = 1/2$ *work. With these share indices, player* 2 *has to pay* $1/3$ *of the cost of local public good production in jurisdiction* $\{1, 2\}$*. Player 2's demand for private and local public good in jurisdiction* $\{1, 2\}$ *is then found as the solution to*

$$\begin{array}{ll} \textit{maximize} & x_2 + y \\ \textit{subject to} & \frac{1}{3}(4y) + x_2 = 2 \\ & x_2 \geq 0,\ y \geq 0 \end{array}$$

The solution to this maximization problem is $\overline{y} = 0$ *and* $\overline{x}_2 = 2$*, which shows that player 2 cannot get a higher utility when a positive amount of local public good is produced in jurisdiction* $\{1, 2\}$*. It follows straightforwardly that player 1, who has to pay a larger share of the cost of local public good production in jurisdiction* $\{1, 2\}$ *than player 2 has to pay, also cannot get a higher utility when a positive amount of local public good is produced in jurisdiction* $\{1, 2\}$*.*

So, the share indices $s_1 = 1$ *and* $s_2 = 1/2$ *work. Of course, equal shares work too. In fact, fixing* $s_1 = 1$*, it can be shown quite easily that 2's share index can be anything between 1/3 and 1 and that the bound 1/3 is sharp. Also, because of symmetry between the two players, we know that all that matters is the* ratio *between the two share indices, so that we can conclude* *that with $s_1 = 1$ any s_2 between 1 and 3 also works.*

Examples 1 and 2 demonstrate that symmetric players do not necessarily have the same share index in a share equilibrium. However, for an economy that satisfies Assumption 1 it holds that in every share equilibrium symmetric players have the same utility, even when their share indices are different.

Theorem 5 *Let E be an economy that satisfies Assumption 1 and let i and j be symmetric players in E. Let $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))$ be a share equilibrium in economy E. Then $u_i(x_i, y_{P(i)}, P(i)) = u_j(x_j, y_{P(j)}, P(j))$.*

Proof. If the symmetric players i and j have the same share index, then they have the same options under condition 2 of the share equilibrium, from which it follows easily that the symmetric players have the same utility in the share equilibrium. Thus, by Theorems 3 and 4, we only have to cover the cases in which the symmetric players are each in a jurisdiction by themselves or in which they are together in a jurisdiction in which no local public good is produced.

Case 1: Suppose $P(i) = \{i\}$ and $P(j) = \{j\}$. Then player i pays the entire cost of local public good production in jurisdiction $P(i)$ (because $s_i^{P(i)} = 1$) and, similarly, player j pays the entire cost of local public good production in jurisdiction $P(j)$. Hence,

$$\begin{aligned} s_i^{P(i)} c\left(y_{P(j)}, P(i)\right) + x_j &= c\left(y_{P(j)}, P(i)\right) + x_j \\ &= c\left(y_{P(j)}, P(j)\right) + x_j \\ &= s_j^{P(j)} c\left(y_{P(j)}, P(j)\right) + x_j \\ &= w_j = w_i, \end{aligned}$$

where the first and third equalities follow from the definition of the relative shares, the second and fifth equalities follow from the symmetry of players i and j, and the fourth equality follows from condition 1 of the share equilibrium. Similarly, we derive that

$$s_j^{P(j)} c\left(y_{P(i)}, P(j)\right) + x_i = w_j.$$

This means that the players can copy each other's consumption patterns, so that condition 2 of the share equilibrium implies

$$\begin{aligned} u_i\left(x_i, y_{P(i)}, P(i)\right) &\geq u_i\left(x_j, y_{P(j)}, P(i)\right) \\ &= u_j\left(x_j, y_{P(j)}, P(j)\right) \\ &\geq u_j\left(x_i, y_{P(i)}, P(j)\right) \\ &= u_i\left(x_i, y_{P(i)}, P(i)\right), \end{aligned}$$

where both equalities follow from the symmetry of players i and j. Because the first and the last expression in this sequence are the same, all weak inequalities, in fact, hold with equality. Thus, $u_i\left(x_i, y_{P(i)}, P(i)\right) = u_j\left(x_j, y_{P(j)}, P(j)\right)$.

Case 2: Suppose $P(i) = P(j)$ and $y_{P(i)} = 0$. Because $c(0, J) = 0$, it follows using condition 1 of the share equilibrium and symmetry of players i and j that

$$\begin{aligned} x_i &= w_i - s_i^{P(i)} c\left(y_{P(i)}, P(i)\right) \\ &= w_i - 0 = w_j - 0 \\ &= w_j - s_j^{P(j)} c\left(y_{P(j)}, P(j)\right) = x_j. \end{aligned}$$

Thus, players i and j belong to the same jurisdiction and consume the same amounts of private and local public goods. Symmetry then tells us that they have the same utility. ■

6 Existence of share equilibrium in symmetric economies

The insights on symmetric players in the previous section can be used to derive a share equilibrium existence result for economies in which all players are symmetric. In such an economy, the players' identities do not matter for costs of local public good production in jurisdictions or in other players' preferences for fellow jurisdiction members. This necessarily implies that costs for producing the local public good in a jurisdiction only depend on *how many* players are in the jurisdiction and also that players care only about *how many* players are in their jurisdiction. This allows simplified notation.

Definition 5 *A symmetric economy is an economy in which all players are symmetric. Precisely, a local public good economy* $E = \langle N; (w_i)_{i \in N}; (u_i)_{i \in N}; c \rangle \in \mathcal{E}$ *is* symmetric *if there exist:*

(a) *a* $w \in \mathbb{R}_+$ *such that* $w_i = w$ *for all* $i \in N$*,*

(b) *a function* $c : \mathbb{R}_+ \times \mathbb{N} \to \mathbb{R}_+$ *such that* $c(y, J) = c(y, |J|)$ *for all* $y \in \mathbb{R}_+$ *and* $J \subseteq N$*,*[20] *and*

(c) *a function* $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{N} \to \mathbb{R}$ *such that for each player* $i \in N$*, each jurisdiction* $J \subseteq N$ *with* $i \in J$*, each amount of private good* x_i *and amount of local public good* y*, it holds that* $u_i(x_i, y, J) = u(x_i, y, |J|)$*.*

We denote the set of symmetric local public good economies in $\mathcal{E}$ *by* $\mathcal{ES}$*, and we denote a symmetric local public good economy* $E \in \mathcal{ES}$ *by* $E = \langle N; w; u; c \rangle$*, with notations as above.*

Taking together the facts demonstrated in Theorems 3, 4, and 5, as well as Examples 1 and 2, we obtain the following result for symmetric economies.

Theorem 6 *Let* E *be a symmetric economy that satisfies Assumption 1 and let* $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))$ *be a share equilibrium in economy* E*. Then (at least) one of the following is true.*

1. *All players have the same share index.*

[20] $\mathbb{N}$ denotes the set of positive integers and, for any finite set J, $|J|$ denotes the number of elements of J.

2. *There is one jurisdiction including all players (i.e.,* $\mathbf{P} = \{N(E)\}$*) and no public good is produced in this jurisdiction* ($y_{N(E)} = 0$*).*

3. *Each player is in a singleton jurisdiction (i.e.,* $P(i) = \{i\}$ *for all* $i \in N(E)$*).*

Moreover, all players have the same utility in the share equilibrium.

In Theorem 6, options 2 and 3 are clearly mutually exclusive. Note, however, that options 2 or 3 do not preclude option 1. The theorem spells out the very special circumstances under which the players in a share equilibrium in a symmetric economy can possibly have different share indices. Basically, we can only find share equilibria in which players have different share indices if there is no public good being produced - either because there is only one jurisdiction and no public good is being produced in that jurisdiction, or because each player forms a singleton jurisdiction and thus y is really a private good in each such jurisdiction.

We can use Theorem 6 to derive conditions that are necessary and sufficient for the existence of a share equilibrium in symmetric local public good economies. To do this, we need the following assumptions.

Assumption 2 *We formulate this assumption for a general economy* $E = \langle N; (w_i)_{i \in N}; (u_i)_{i \in N}; c \rangle \in \mathcal{E}$.

a. *For each player* $i \in N$ *and each jurisdiction* $J \subseteq N$ *with* $i \in J$*, it holds that the utility function* $u_i(\cdot, \cdot, J)$ *is strictly increasing and continuous (as a function of* x_i *and* y*, keeping the jurisdiction fixed).*

b. *For each jurisdiction* $J \subseteq N$*, it holds that the cost function* $c(\cdot, J)$ *is non-decreasing and continuous (as a function of* y*, keeping the jurisdiction fixed), and* $c(0, J) = 0$*.*

c. *For each jurisdiction* $J \subseteq N$*, there exists a level of public good* y *such that* $c(y, J) > \sum_{i \in J} w_i$*.*

Note that the basic assumptions on local public good economies are subsumed in parts a and b of Assumption 2, and that these two parts are otherwise weaker than the corresponding parts of Assumption 1. Part c of Assumption 2 merely implies that the cost function is such that no jurisdiction can afford an unlimited level of local public good and it is a very weak assumption.

Let $E = \langle N; w; u; c\rangle \in \mathcal{ES}$ be a symmetric economy that satisfies Assumption 2. Also, let $k \in \mathbb{N}$ be an arbitrary jurisdiction size. There exists a largest level of local public good $y(k)$ satisfying $c(y(k), k) = kw$; this follows from the assumptions that (i) the cost function c is continuous, (ii) $c(0, k) = 0$, (iii) c is non-decreasing in y, and (iv) there exists a level of local public good y such that $c(y, k) > kw$ (which follows from part c of Assumption 2 applied to the symmetric economy E). We define a level of local public good

$$y_k \in \arg \max_{y \in [0, y(k)]} k \cdot u \left(w - \frac{1}{k} c\left(y, k\right), y, k \right).$$

Since the set $[0, y(k)]$ is compact and u and c are continuous, such a y_k exists. The quantity y_k is a level of local public good that maximizes the sum of the utilities of the players in a jurisdiction of size k when the players in the jurisdiction share the costs of local public good production equally. We denote this maximum total utility for a jurisdiction of size k by $U(k)$;

$$U(k) = k \cdot u \left(w - \frac{1}{k} c\left(y_k, k\right), y_k, k \right).$$

Now, for each coalition of players $S \subseteq N$, we define $v^E(S)$ to be the maximum total utility obtainable by the players in a coalition S when they consider forming various jurisdictions not including any players not in S and when the costs of local public good production are shared equally in each possible jurisdiction;

$$v^E(S) = \max_{\mathbf{P} \in \mathcal{P}(S)} \sum_{J \in \mathbf{P}} U(|J|), \tag{10}$$

where $\mathcal{P}(S)$ denotes the set of all partitions of S. Note that a coalition of players may have to split up into multiple jurisdictions to obtain the maximal utility $v^E(S)$ for its members.

Definition 6 *Economy E is* top convex *if the function v^E satisfies the condition that $\frac{v^E(S)}{|S|} \leq \frac{v^E(N)}{|N|}$ for all $S \subseteq N$.*

Top convexity means that, among all configurations in the economy that are symmetric within each of their jurisdictions, the per-capita utility of the players in the economy is maximal in some configuration that maximizes the sum of all players' utilities.[21] Note that the condition of top convexity allows congestion; it is possible that $v^E(N)$ is achieved by some non-trivial partition of players into jurisdictions. Informally, top convexity implies that the player set N can be partitioned into 'optimal' jurisdictions. We illustrate the top convexity condition in the following example. The example is chosen in part to demonstrate how models in the literature on economies with local public goods, such as Allouch and Wooders (2008) and its antecedents, fit into the framework used in the current paper.

Example 3 *A top convex, symmetric local public good economy.*

Assume that all players are identical in terms of their endowments, preferences, and effects on others. Each player's utility is additively separable in private and local public good consumption on one hand and jurisdiction membership on the other hand. The utility for private and local public good

[21] A related condition that is also called top convexity was defined in a setting of networks in Jackson and van den Nouweland (2005), who in turn got their inspiration from a condition called "domination by the grand coalition" that was defined for coalitional games by Chatterjee et al. (1993). This condition, which stipulates that the per-capita value is maximal for the grand coalition, was identified but unnamed by Shubik (1982, p.149) as a necessary and sufficient requirement for the non-emptiness of the core of a coalitional game with transferable utility. Our condition differs in that, as in much of the local public good literature, we allow for congestion and multiple jurisdictions and consider the maximum per capita utility over all partitions of the set of players into jurisdictions.

consumption takes a Cobb-Douglas form and the utility for jurisdiction membership depends on jurisdiction size, so that it has the interpretation of a congestion or crowding function. The cost of local public good production is taken such that the per capita *cost are constant across levels of local public good and jurisdictions. This is a special case that often appears in the literature and that makes the good a local public service. We may think of the local public good as, for example, some service offered by a condominium for which the costs are fixed in per capita terms.*[22]

Formally, let $E = \langle N; w; u; c\rangle$ *be a symmetric local public good economy with* $u(x_i, y, J) = x_i^{\alpha} y^{1-\alpha} - v(|J|)$, *where* $\alpha \in (0,1)$, $v : \mathbb{N} \to \mathbb{R}$ *is the anonymous congestion or crowding function, and* $c(y, J) = |J|\, y$.[23]

We choose the crowding function v *so that only jurisdictions of size 2 or 3 are desirable:*

$$v(k) = \begin{cases} 0 & \textit{if } k = 2 \textit{ or } k = 3 \\ 1 & \textit{otherwise.} \end{cases}$$

We start by computing y_k *for a jurisdiction consisting of* k *players.*

$$k \cdot u\left(w - \frac{1}{k} c\,(y, k)\,, y, k\right) = k\left((w-y)^{\alpha} y^{1-\alpha} - v(k)\right),$$

where $v(k)$ *is either 0 or 1. The level of local public good that maximizes this expression is equal to* $y_k = (1-\alpha)w$, *which implies that in this example the optimal level of local public good is independent of the size of the jurisdiction.*[24]

We can now compute the maximum total utility $U(k)$ *of the players in a jurisdiction of size* k. *We simplify notation by defining* $A = \alpha^{\alpha}(1-\alpha)^{1-\alpha} w$,

[22] For example, the local public good could be window-cleaning in the condominium. The condominium assocation of one of the authors provides this service from condominium association fees.

[23] Note that with these definitions the economy satisfies the continuity and cost conditions of Assumption 2.

[24] Of course, this is a feature that follows from the way in which we set up this particular example to make it easily tractable. It is not true in general for a symmetric local public good economy.

a quantity that is independent of the size of the jurisdiction.

$$\begin{aligned} U(k) &= k\left((w-(1-\alpha)w)^{\alpha}((1-\alpha)w)^{1-\alpha}-v(k)\right) \\ &= kA - kv(k). \end{aligned}$$

Note that the first component in this expression results in a per-capita utility A that is constant across jurisdictions of various sizes, whereas the second component varies with jurisdiction size on a per-capita basis.

Because of the way in which we defined the crowding function v, it follows that a coalition of players reaches the maximum total utility by breaking itself up into jurisdictions of size 2 or 3. This can be done for any coalition size except for a coalition consisting of a single player: A coalition with an even number n of players can be broken up into $\frac{n}{2}$ jurisdictions of size 2 and a coalition with an odd number $n \geq 3$ of players can be broken up into 1 jurisdiction of size 3 and $\frac{n-3}{2}$ jurisdictions of size 2. Using this, we obtain

$$v^E(S) = \begin{cases} A-1 & \text{if } |S| = 1, \\ |S|A & \text{if } |S| > 1. \end{cases}$$

Thus,

$$\frac{v^E(S)}{|S|} = \begin{cases} A-1 & \text{if } |S| = 1, \\ A & \text{if } |S| > 1, \end{cases}$$

and the economy E is top convex because the maximum per-capita utility can be obtained for the coalition consisting of all players (by splitting itself up into two- or three-person jurisdictions).

Theorem 7 demonstrates that a share equilibrium exists for every top convex symmetric local public good economy with continuous utility and cost functions in which no jurisdiction can afford an unlimited level of local public good.

Theorem 7 *Let $E = \langle N; w; u; c\rangle$ be a symmetric local public good economy satisfying Assumption 2 that is top convex. Then $SE(E) \neq \emptyset$.*

Proof. Define share indices $\mathbf{s}$ by $s_i = 1$ for each $i \in N$, so that all players have the same share index. Also, let

$$P(N) \in \arg\max_{\mathbf{P} \in \mathcal{P}(N)} \sum_{J \in \mathbf{P}} \left(U(|J|)\right).$$

For each jurisdiction $J \in P(N)$, choose a

$$y_J \in \arg\max_{y \in [0, y(|J|)]} |J| \cdot u\left(w - \frac{c\left(y, |J|\right)}{|J|}, y, |J|\right).$$

For each $i \in N$, denote the jurisdiction in $P(N)$ containing player i by $J(i)$ and define a level of private good consumption by

$$x_i = w - \frac{c\left(y_{J(i)}, |J(i)|\right)}{|J(i)|}.$$

We will prove that the share indices $\mathbf{s}$ and the configuration $((x_i)_{i \in N}, (y_J)_{J \in P(N)}, P(N))$ form a share equilibrium in the symmetric economy E.

Claim 1. $\sum_{J \in P(N)} v^E(J) = v^E(N)$. This follows from the following string of (in)equalities.

$$\begin{aligned}
\sum_{J \in P(N)} v^E(J) &= \sum_{J \in P(N)} \left(\max_{\mathbf{P} \in \mathcal{P}(J)} \sum_{K \in \mathbf{P}} \left(|K| \cdot u\left(w - \frac{c\left(y_{|K|}, |K|\right)}{|K|}, y_{|K|}, |K|\right)\right)\right) \\
&\leq \max_{\mathbf{P} \in \mathcal{P}(N)} \sum_{K \in \mathbf{P}} \left(|K| \cdot u\left(w - \frac{c\left(y_{|K|}, |K|\right)}{|K|}, y_{|K|}, |K|\right)\right) \\
&= v^E(N) \\
&= \sum_{J \in P(N)} \left(|J| \cdot u\left(w - \frac{c\left(y_{|J|}, |J|\right)}{|J|}, y_{|J|}, |J|\right)\right) \\
&\leq \sum_{J \in P(N)} \left(\max_{\mathbf{P} \in \mathcal{P}(J)} \sum_{K \in \mathbf{P}} \left(|K| \cdot u\left(w - \frac{c\left(y_{|K|}, |K|\right)}{|K|}, y_{|K|}, |K|\right)\right)\right) \\
&= \sum_{J \in P(N)} v^E(J)
\end{aligned}$$

Claim 2. $v^E(J) = |J| \cdot u\left(w - \frac{c\left(y_{|J|}, |J|\right)}{|J|}, y_{|J|}, |J|\right)$ for all $J \in P(N)$. This follows from the fact that the last (weak) inequality in the sequence of (in)equalities above is an equality, as we have just derived.

Claim 3. $\frac{v^E(J)}{|J|} = \frac{v^E(N)}{|N|}$ for all $J \in P(N)$. To see this, we derive

$$v^E(N) = \sum_{i \in N} \frac{v^E(N)}{|N|} = \sum_{J \in P(N)} \sum_{i \in J} \frac{v^E(N)}{|N|} = \sum_{J \in P(N)} |J| \frac{v^E(N)}{|N|} \geq \sum_{J \in P(N)} v^E(J),$$

where the last inequality follows because top convexity of E implies that $|J| \frac{v^E(N)}{|N|} \geq v^E(J)$ for each $J \in P(N)$. Because $\sum_{J \in P(N)} v^E(J) = v^E(N)$ by Claim 1, it follows that all the weak inequalities are in fact equalities, so that we can derive that $|J| \frac{v^E(N)}{|N|} = v^E(J)$ for each $J \in P(N)$.

We are now ready to prove that $\left(\mathbf{s}, ((x_i)_{i \in N}, (y_J)_{J \in P(N)}, P(N))\right)$ is a share equilibrium. First note that for each potential jurisdiction the costs of local public good production are shared equally among all jurisdiction members because all players have the same share index. Condition 1 of the share equilibrium then follows from $x_i = w - \frac{c\left(y_{J(i)}, |J(i)|\right)}{|J(i)|}$. To show that condition 2 of the share equilibrium also holds, fix $i \in N$ and let $(\overline{x}_i, \overline{y}, \overline{J}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times 2^N$ be such that $i \in \overline{J}$ and $\frac{c(\overline{y}, |\overline{J}|)}{|\overline{J}|} + \overline{x}_i \leq w$. We need to show that $u_i(x_i, y_{J(i)}, J(i)) \geq u_i(\overline{x}_i, \overline{y}, \overline{J})$. First, note that $c(\overline{y}, |\overline{J}|) \leq |\overline{J}| \, (w - \overline{x}_i) \leq |\overline{J}| w$, from which it follows that $\overline{y} \leq y\left(|\overline{J}|\right)$. Using this, we derive

$$
\begin{aligned}
u_i\left(\overline{x}_i, \overline{y}, \overline{J}\right) &\leq u\left(w - \frac{c(\overline{y}, |\overline{J}|)}{|\overline{J}|}, \overline{y}, |\overline{J}|\right) \\
&\leq \frac{1}{|\overline{J}|}\left(|\overline{J}| \cdot u\left(w - \frac{c(\overline{y}_{|\overline{J}|}, |\overline{J}|)}{|\overline{J}|}, \overline{y}_{|\overline{J}|}, |\overline{J}|\right)\right) \\
&\leq \frac{1}{|\overline{J}|} v^E(\overline{J}) \\
&\leq \frac{v^E(N)}{|N|} \\
&= \frac{v^E(J(i))}{|J(i)|} \\
&= u\left(w - \frac{c\left(y_{|J(i)|}, |J(i)|\right)}{|J(i)|}, y_{|J(i)|}, |J(i)|\right) \\
&= u\left(w - \frac{c\left(y_{J(i)}, |J(i)|\right)}{|J(i)|}, y_{J(i)}, |J(i)|\right) \\
&= u_i(x_i, y_{J(i)}, J(i)),
\end{aligned}
$$

where the first inequality follows from utility being strictly increasing in private good consumption, the second inequality follows from the definition of $\overline{y}_{|\overline{J}|}$, the third inequality follows from the definition of $v^E(\overline{J})$, the fourth inequality follows from top convexity of E, the first equality follows by Claim 3, the second equality follows from Claim 2, the third equality follows from the definition of $y_{J(i)}$, and the last equality follows from the definition of x_i.
■

We illustrate Theorem 7 in the following example.

Example 4 *A share equilibrium in a top convex symmetric economy.*

Consider the top convex symmetric local public good economy in example 3. We remind the reader that for each $k \in \mathbb{N}$ we computed $y_k = (1-\alpha)w$ and

$$U(k) = kA - kv(k),$$

where $A = \alpha^\alpha(1-\alpha)^{1-\alpha}w$. We also determined that a coalition of players reaches the maximum total utility by breaking itself up into jurisdictions of *size 2 or 3. Because the one-player economy is uninteresting, assume that E has at least two players.*

The share equilibria of this economy identified in Theorem 7 consist of a share index $s_i = 1$ for each player i and a configuration $(\mathbf{x}, \mathbf{y}, \mathbf{P})$, where $\mathbf{P}$ is a partition of $N(E)$ into jurisdictions of size 2 or 3, $y_J = (1-\alpha)w$ for each $J \in \mathbf{P}$, and

$$x_i = w - \frac{c\left(y_{J(i)}, J(i)\right)}{|J(i)|} = \alpha w$$

for each player i. Note that we are identifying multiple share equilibrium configurations. For example, if there are 6 players, then we have identified share equilibria with each of the 15 possible partitions of $N(E)$ into 3 jurisdictions of size 2 and also with each of the 10 possible partitions of $N(E)$ into 2 jurisdictions of size 3.

Theorem 7 demonstrates that, under the mild Assumption 2, top convexity is a sufficient condition for existence of share equilibrium in symmetric economies. The following theorem establishes that top convexity is also necessary for the existence of share equilibrium in symmetric economies when Assumption 1 is added.

Theorem 8 *Let $E = \langle N; w; u; c\rangle$ be a symmetric local public good economy satisfying Assumptions 1 and 2, and suppose that $SE(E) \neq \emptyset$. Then E is top convex.*

Proof. Let $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))$ be a share equilibrium in economy E. By Theorem 1 we know that the configuration $(\mathbf{x}, \mathbf{y}, \mathbf{P})$ is in the core of E. Theorem 6 implies that all players have the same utility in the share equilibrium and that there are only two cases in which the players' share indices $\mathbf{s}$ can potentially be different. We discuss the three cases in Theorem 6 in turn.

Case 1. All players have the same share index. Without loss of generality, we assume that $s_i = 1$ for all $i \in N$. We can interpret the function v^E as defined in (10) as the characteristic function of a symmetric coalitional game (N, v^E) that is obtained using the equal share indices $\mathbf{s}$.[25] Because all players have the same utility in the share equilibrium, we know that there exists a $\tilde{u}$ such that $u(x_i, y_{P(i)}, P(i)) = \tilde{u}$ for all $i \in N$. Using that $(\mathbf{x}, \mathbf{y}, \mathbf{P})$ is in the core of E, we know that the vector of the players' utilities $\big(u(x_i, y_{P(i)}, P(i))\big)_{i\in N} = (\tilde{u})_{i\in N}$ is in the core of the coalitional game (N, v^E). Thus, $(\tilde{u})_{i\in N}$ is a symmetric core element of the symmetric coalitional game (N, v^E). This implies that $\tilde{u} = \frac{v^E(N)}{|N|}$ and $\sum_{i\in S} \tilde{u} \geq v^E(S)$ for all $S \subseteq N$. Thus, for all $S \subseteq N$ it holds that $|S| \frac{v^E(N)}{|N|} \geq v^E(S)$, which is top convexity of E.

Case 2. There is one jurisdiction including all players (i.e., $\mathbf{P} = \{N\}$) and no public good is produced in this jurisdiction ($y_N = 0$). Give every player the same share index: $\hat{s}_i = 1$ for all $i \in N$. We will show that $(\hat{\mathbf{s}}, (\mathbf{x}, 0, \{N\}))$ is a share equilibrium in economy E. Note that this suffices to prove that E is top convex because of Case 1.

Condition 1 of share equilibrium is satisfied in $(\hat{\mathbf{s}}, (\mathbf{x}, 0, \{N\}))$ because for each player $i \in N$ we have that $\hat{s}_i^N c(0, N) + x_i = 0 + x_i = s_i^N c(0, N) + x_i = w$, where the last equality follows because $(\mathbf{s}, (\mathbf{x}, 0, \{N\})) \in SE(E)$. Note that this also immediately implies that $x_i = w$ for all $i \in N$. To verify that condition 2 of share equilibrium is satisfied, let $i \in N$ and $(\overline{x}_i, \overline{y}, \overline{J}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times 2^N$ be such that $i \in \overline{J}$ and $\hat{s}_i^{\overline{J}} c(\overline{y}, \overline{J}) + \overline{x}_i \leq w$. We will show that player i does not prefer $(\overline{x}_i, \overline{y}, \overline{J})$ to $(x_i, 0, \{N\})$.

Note that $\hat{s}_j^{\overline{J}} = \frac{1}{|\overline{J}|}$ for all players $j \in \overline{J}$. Define $j \in \overline{J}$ to be the player with the lowest relative cost share in $\overline{J}$ according to the share indices $\mathbf{s}$, so that $s_j^{\overline{J}} \leq \frac{1}{|\overline{J}|}$. It follows that $s_j^{\overline{J}} c(\overline{y}, \overline{J}) + \overline{x}_i \leq \hat{s}_i^{\overline{J}} c(\overline{y}, \overline{J}) + \overline{x}_i \leq w$. Thus, with share indices $\mathbf{s}$, player j can afford to consume $\overline{x}_i$ of the private good and $\overline{y}$ of the public good in jurisdiction $\overline{J}$. It thus follows by condition 2 of share equilibrium applied to $(\mathbf{s}, (\mathbf{x}, 0, \{N\})) \in SE(E)$ and player j with utility function u, that $u(x_j, 0, \{N\}) \geq u(\overline{x}_i, \overline{y}, \overline{J})$. Using $x_i = w = x_j$, we

[25] We refer readers who are not familiar with coalitional games and their cores to Moulin (1995), Owen (2013), or any other text covering basic cooperative game theory.

then verify that $u(x_i, 0, \{N\}) \geq u(\overline{x}_i, \overline{y}, \overline{J})$, so that player i, who also has utility function u, does not prefer $(\overline{x}_i, \overline{y}, \overline{J})$ to $(x_i, 0, \{N\})$.

Case 3. Each player is in a singleton jurisdiction (i.e., $P(i) = \{i\}$ for all $i \in N$). Give every player the same share index: $\hat{s}_i = 1$ for all $i \in N$. We will show that $(\hat{\mathbf{s}}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))$ is a share equilibrium in economy E. Note that this suffices to prove that E is top convex because of Case 1.

Condition 1 of share equilibrium is satisfied in $(\hat{\mathbf{s}}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))$ because for each player $i \in N$ we have that $P(i) = \{i\}$ and thus $\hat{s}_i^{P(i)} = s_i^{P(i)} = 1$, so that $\hat{s}_i^{P(i)} c(y_{P(i)}, P(i)) + x_i = s_i^{P(i)} c(y_{P(i)}, P(i)) + x_i = w$. To verify that condition 2 of share equilibrium is also satisfied, let $i \in N$ and $(\overline{x}_i, \overline{y}, \overline{J}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times 2^N$ be such that $i \in \overline{J}$ and $\hat{s}_i^{\overline{J}} c(\overline{y}, \overline{J}) + \overline{x}_i \leq w$. We will show that player i does not prefer $(\overline{x}_i, \overline{y}, \overline{J})$ to $(x_i, y_{\{i\}}, \{i\})$.

Note that $\hat{s}_j^{\overline{J}} = \frac{1}{|\overline{J}|}$ for all players $j \in \overline{J}$. Define $j \in \overline{J}$ to be the player with the lowest relative cost share in $\overline{J}$ according to the share indices $\mathbf{s}$, so that $s_j^{\overline{J}} \leq \frac{1}{|\overline{J}|}$. It follows that $s_j^{\overline{J}} c(\overline{y}, \overline{J}) + \overline{x}_i \leq \hat{s}_i^{\overline{J}} c(\overline{y}, \overline{J}) + \overline{x}_i \leq w$. Thus, with share indices $\mathbf{s}$, player j can afford to consume $\overline{x}_i$ of the private good and $\overline{y}$ of the public good in jurisdiction $\overline{J}$ and it follows by condition 2 of share equilibrium applied to the share equilibrium $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))$ and player j with utility function u, that $u(x_j, y_{\{j\}}, \{j\}) \geq u(\overline{x}_i, \overline{y}, \overline{J})$. Using that we know that all players have the same utility in the share equilibrium $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))$, we can thus conclude that $u(x_i, y_{\{i\}}, \{i\}) = u(x_j, y_{\{j\}}, \{j\}) \geq u(\overline{x}_i, \overline{y}, \overline{J})$. This establishes that player i does not prefer $(\overline{x}_i, \overline{y}, \overline{J})$ to $(x_i, y_{\{i\}}, \{i\})$. ■

Theorems 7 and 8 combined immediately lead to a characterization of symmetric economies that allow share equilibria.

Theorem 9 *Let $E = \langle N; w; u; c \rangle$ be a symmetric local public good economy satisfying Assumptions 1 and 2. Then E admits a share equilibrium if and only if E is top convex.*

The statement in Theorem 9 can be strengthened somewhat by addressing possible variation in equilibrium share indices. In Theorem 7, we identified a share equilibrium in which all players have the same share index.

While this demonstrates existence of share equilibrium, it does not address if other share indices might give rise to alternative equilibrium configurations. However, the proof of Theorem 8 reveals that in a symmetric economy satisfying Assumptions 1 and 2, every equilibrium configuration can be supported by equal share indices for all players.

Readers familiar with the game theory result that for symmetric games top convexity is a necessary and sufficient condition for non-emptiness of the core, may at this point wonder whether a similar result can be derived for symmetric local public good economies. In light of Theorem 9, we would then obtain a result that a symmetric local public good economy satisfying Assumptions 1 and 2 admits a share equilibrium if and only if the core of the economy is non-empty. One of these implications is a direct corollary of Theorem 1, which states that every equilibrium configuration is in the core of an economy. The other implication, however, does not hold. This is because a symmetric local public good economy with a non-empty core may not have any symmetric core configurations, whereas we have demonstrated in Theorem 6 that all players have the same utility in a share equilibrium. To see that a symmetric local public good economy with a non-empty core may not have any symmetric core configurations, it is important to note that Assumptions 1 and 2 have very little bite when it comes to variation in players' preferences across jurisdictions of different sizes, for the simple reason that jurisdiction sizes are natural numbers and thus the continuity assumptions put no restrictions on this dimension of players' preferences. In addition, none of our assumptions rule out the possibility that in the grand coalition (or, for that matter, in any other jurisdiction) the marginal utility for private good consumption is increasing in the level of private good, for a fixed level of public good. Thus, it is possible to have a core configuration where the grand coalition N is formed and a level y of the public good is consumed, and the cost $c(y, N)$ is distributed unevenly across the players. An even distribution of that cost among the players does not give rise to a core configuration if there exists some smaller jurisdiction in which all members can get a higher utility when they equally distribute the costs of a particular level of local public good among themselves.

If we are willing to make assumptions that link players' preferences in jurisdictions of varying sizes in such a way that they prevent the existence of core configurations in which the grand coalition is formed, then a symmetric local public good economy with a non-empty core will have a symmetric core configuration.[26] Moreover, such a symmetric core configuration is an equilibrium configuration and thus we get existence of share equilibrium in such economies.

Theorem 10 *Let $E = \langle N; w; u; c\rangle$ be a symmetric local public good economy satisfying Assumptions 1 and 2, and let configuration $(\mathbf{x}, \mathbf{y}, \mathbf{P})$ be in the core of E and $\mathbf{P} \neq \{N\}$. Then $SE(E) \neq \emptyset$.*

Proof. We first prove that for the core configuration $(\mathbf{x}, \mathbf{y}, \mathbf{P})$, it holds that $u(x_i, y_{P(i)}, P(i)) = u(x_j, y_{P(j)}, P(j))$ for all players $i, j \in N$. This can be seen as follows: Let $(\overline{\mathbf{x}}, \overline{\mathbf{y}}, \mathbf{P})$ be a configuration in E in which not all players have the same utility. Because $\mathbf{P} \neq \{N\}$, we can choose two players i and j that are not in the same jurisdiction (i.e., $j \notin P(i)$) and such that $u(\overline{x}_i, \overline{y}_{P(i)}, P(i)) \neq u(\overline{x}_j, \overline{y}_{P(j)}, P(j))$. Without loss of generality, we assume that $u(\overline{x}_i, \overline{y}_{P(i)}, P(i)) > u(\overline{x}_j, \overline{y}_{P(j)}, P(j))$. Then player j can replace player i in jurisdiction $P(i)$ and this will result in a higher utility for player j and an unchanged utility for the remaining players in $P(i)$ (if there are any). Thus, the configuration $(\overline{\mathbf{x}}, \overline{\mathbf{y}}, \mathbf{P})$ is not in the core of E.

Because all players have the same utility in the configuration $(\mathbf{x}, \mathbf{y}, \mathbf{P})$, we know that there exists a $\tilde{u}$ such that $u(x_i, y_{P(i)}, P(i)) = \tilde{u}$ for all $i \in N$. As we did in the proof Theorem 8 (Case 1), we interpret the function v^E

[26] One such assumption is strict small group effectiveness, an assumption that was introduced for economies with local public goods in Wooders (1978) and variants of which play a key role in results on nonemptiness of approximate cores of games with many players. See, for example, Kovalenkov and Wooders (2003).

as defined in (10) as the characteristic function of a symmetric coalitional game (N, v^E). Using that $(\mathbf{x}, \mathbf{y}, \mathbf{P})$ is in the core of E, we know that the vector of the players' utilities $\left(u(x_i, y_{P(i)}, P(i))\right)_{i\in N} = (\tilde{u})_{i\in N}$ is in the core of the coalitional game (N, v^E). Thus, $(\tilde{u})_{i\in N}$ is a symmetric core element of the symmetric coalitional game (N, v^E). This implies that $\tilde{u} = \frac{v^E(N)}{|N|}$ and $\sum_{i\in S} \tilde{u} \geq v^E(S)$ for all $S \subseteq N$. Thus, for all $S \subseteq N$ it holds that $|S| \frac{v^E(N)}{|N|} \geq v^E(S)$, which demonstrates top convexity of E. Now, it follows from Theorem 7 that $SE(E) \neq \emptyset$. ■

References

[1] Allouch, N. and M. Wooders (2008) "Price taking equilibrium in economies with multiple memberships in clubs and unbounded club sizes," *Journal of Economic Theory*, doi:10,1016/j.jet.2007.07.06.

[2] Chatterjee, K., B. Dutta, D. Ray, and S. Sengupta (1993) "A noncooperative theory of coalitional bargaining,"*Review of Economic Studies* 60, 463-477.

[3] Debreu G. and Scarf, H. (1963) "A limit theorem on the core of an economy," *International Economic Review* 4, 235-246.

[4] Foley, D. (1970) "Lindahl's solution and the core of an economy with public goods," *Econometrica* 38, 66-72.

[5] Jackson, M., and A. van den Nouweland (2005) "Strongly stable networks," *Games and Economic Behavior* 51 (2), 420-444.

[6] Kaneko, M. (1977) "The ratio equilibria and the core of the voting game G(N,w) in a public goods economy," *Econometrica* 45, 1589-1594.

[7] Kovalenkov, A. and M. Wooders (2003) "Approximate cores of games and economies with clubs," *Journal of Economic Theory* 110, 87-120.

[8] Lindahl, E. (1919) "Just taxation - a positive solution." In: *Classics in the Theory of Public Finance* (Eds. R. Musgrave and A. Peacock) (1958), Macmillan, London, United Kingdom.

[9] Moulin, H. (1995) *Cooperative Microeconomics; A Game-Theoretic Introduction*, Prentice Hall/Harvester Wheatsheaf, Hertfordshire, United Kingdom.

[10] Owen, G. (2013) *Game Theory*, Emerald Group Publishing, Bingley, United Kingdom.

[11] Shubik, M. (1982) *Game Theory in the Social Sciences: Concepts and Solutions*, MIT Press, Cambridge, MA.

[12] Van den Nouweland, A., and M. Wooders (2011) "Share equilibrium in local public good economies," *Journal of Mathematical Economics* 47 (3), 376-381.

[13] Wooders, M. (1978) "Equilibria, the core and jurisdiction structures in an economy with a local public good," *Journal of Economic Theory* 18, 328-348.

[14] Wooders, M. (1983) "The epsilon core of a large replica game," *Journal of Mathematical Economics* 11, 277-300.