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Comparing the nucleolus and the Shapley value of 3-player transferable utility games

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Abstract

We propose a simplified version of the formulas defining the nucleolus of a 3-player transferable utility game proposed by Legros (1981) and study its relation to the Shapley value.

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1. Introduction

Cooperative games with transferable utility focus on allocating the worth of the grand coalition among players. A variety of solution concepts have been introduced for this purpose. Some, such as the core, define sets of allocations that satisfy certain stability properties. Others, like the Shapley value and the nucleolus, specify a unique allocation that meets certain criteria. In this paper, we concentrate on the core, the nucleolus, and the Shapley value.

Belonging to the core guarantees stability: no coalition of players, nor any single player, can profitably object to an allocation. However, the core may be empty and, when it is not, stability alone does not ensure fairness. In practice, the core often highlights the underlying competitive pressures present in the underlying game, disadvantaging players in weaker positions.[1] The Shapley value is defined by simple axioms and allocates to a player an amount that exclusively depends upon his contributions. The nucleolus is "*the result of an arbitrator's desire to minimize the dissatisfaction of the most dissatisfied coalition*" to quote Maschler et al. (1979). Yet, the Shapley value does not necessarily yield a core allocation when the core is nonempty, while the nucleolus always produces a core allocation.

Both the Shapley value and the nucleolus are always welldefined. The Shapley value is given by a simple closed-form formula while computing the nucleolus is generally more challenging. However, Legros (1981) provides a formula for 3-player superadditive games.[2] Here, we focus on superadditive games and provide an easily readable formula, following the approach of Dehez and Pacini (2024), who studied the conditions under which the Shapley value belongs to the core. We identify conditions under which the Shapley value and the nucleolus coincide and examine how they differently treat players.

The coincidence between the two solutions is not a new question. It has been addressed as a general question by Kar et al. (2009), Chang and Tseng (2011), González-Díaz and Sánchez-Rodríguez (2014) and Yokote et al. (2017). Here, we restrict ourselves to 3-player games, the distinguishing feature of the present contribution lying in the approach used to address the question. We show that coincidence occurs on a set of measure zero and we provide simple conditions allowing a characterization of the differences between the two solutions in terms of the distribution of payoffs that they induce among the agents.

The paper is organized as follows. Section 2 introduces transferable utility games. Section 3 presents the formula for the nucleolus of a 3-player normalized game. Section 4 investigates the conditions under which the Shapley value and the nucleolus coincide. The differential treatment of the players under the Shapley value and the nucleolus is the object of Section 5. Section 6 offers two applications that illustrate the coincidence and differential treatment results. The last section provides concluding remarks.

2. Transferable utility games

Given a set of players $N = \{1, ..., n\}$, a transferable utility game is defined by a "characteristic" function $v : 2^N \to \mathbb{R}$ that associates a real number $v(S)$ to each coalition $S \subset N$ that measures what coalition S is able to generate on its own. By convention, $v(\emptyset) = 0$.

Notation: Finite sets are denoted by upper-case letters and lower-case letters are used to denote their cardinals: $t = |T|$, $s = |S|$, ... For vector x, $x(S)$ denotes the sum of its coordinates over the subset S. Set inclusion is denoted by $\subset$ and strict inclusion is denoted by $\subsetneq$.

[1] The gloves market (Shapley 1955) well illustrates this feature.

[2] Legros' formula is reproduced in Moulin (1988).

We restrict our attention to the class of superadditive and essential games. A game is *superadditive* if, for any pair of disjoint coalitions S and T, $v(S) + v(T) \leq v(S \cup T)$. In particular, $\sum_{i \in N} v(i) \leq v(N)$. If the latter inequality holds with a strict inequality, the game is said to be *essential*. *Convex* games (Shapley 1971) are games for which the inequality $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ holds for all coalitions S and T. Convexity implies superadditivity. A game is convex if (and only if) marginal contributions are increasing with coalition size. Under superadditivity, the formation of the grand coalition is desirable, and the question then concerns the allocation of the resulting value among players. Ensuring that no player ends up with an amount smaller than his stand-alone worth defines the *imputation set*: $I(N, v) = \{x \in \mathbb{R}^n \mid x(N) = v(N), x_i \geq v(i)\ for\ all\ i \in N\}$.

Any essential game (N, v) can be transformed into a normalized game $(N, \bar{v})$ in which the values of singletons are equal to 0 and the value of the grand coalition is equal to 1:

$$\bar{v}(S) = \frac{v(S) - \sum_{i \in S} v(i)}{v(N) - \sum_{i \in N} v(i)}.$$

Superadditivity and convexity are preserved through normalization. In what follows, we limit to superadditive and essential games involving three players. Its normalized version is then defined by three numbers: $a = \bar{v}(1,2)$, $b = \bar{v}(1,3)$ and $c = \bar{v}(2,3)$. Without loss of generality, we assume that $a \leq b \leq c$. The inequalities $0 \leq a \leq b \leq c \leq 1$ then ensure superadditivity and $P = \{(a, b, c) \in \mathbb{R}^3 \mid 0 \leq a \leq b \leq c \leq 1\}$ is the set of admissible parameters. Convexity requires, moreover, that $a + b \leq 1$, $a + c \leq 1$ and $b + c \leq 1$. Only the latter is actually active, the other two resulting from superadditivity. As a result, games whose parameter c is smaller than 0.5 are convex: $b + c \leq 0.5 + 0.5 = 1$.

Harsanyi (1959) defines the concept of *dividend* associated with each coalition. They are defined recursively, starting with $\alpha_\emptyset^v = 0$, as follows:

$$\alpha_T^v = v(T) - \sum_{S \subsetneq T} \alpha_S^v. \tag{1}$$

The worth of a coalition can then be written as $v(S) = \sum_{T \subset N} \alpha_T^v$ for all $S \subset N$.

Positive games are games whose dividends are non-negative. Positive games are convex.[3] The dividends that are associated with a normalized 3-player game $(N, \bar{v})$ are given by $\alpha^{\bar{v}} = (0, 0, 0|a, b, c|1 - a - b - c)$. Hence, positivity holds if $a + b + c \leq 1$.

3. The Shapley value and nucleolus of a 3-player game

The Shapley value (Shapley 1953) and the nucleolus (Schmeidler 1969) are two prominent allocation rules. Harsanyi (1959) shows that the *Shapley value* allocates to each player the sum of the *per capita* dividends of the coalitions of which he is a member:

$$SV_i(N, v) = \sum_{T \subset N:\ i \in T} \frac{1}{t} \alpha_T^v.$$

Given (1), the Shapley value of a 3-player normalized game $(N, \bar{v})$ is then easily computed:

$$\begin{aligned} SV_1(N, \bar{v}) &= \frac{a+b}{2} + \frac{1-a-b-c}{3} = \frac{a+b}{6} + \frac{1-c}{3}, \\ SV_2(N, \bar{v}) &= \frac{a+c}{2} + \frac{1-a-b-c}{3} = \frac{a+c}{6} + \frac{1-b}{3}, \\ SV_3(N, \bar{v}) &= \frac{b+c}{2} + \frac{1-a-b-c}{3} = \frac{b+c}{6} + \frac{1-a}{3}. \end{aligned} \tag{2}$$

[3] Moreover, solution concepts tend to converge on positive games. For details, see Dehez (2017).

The core extends the idea of imputations from individual players to coalitions:

$$C(N, v) = \{x \in \mathbb{R}^n \mid x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } S \subset N\}.$$

It was introduced by Gillies (1953) and established as a solution concept by Shapley (1955). Bondareva (1963) and Shapley (1967) have independently given necessary and sufficient conditions under which the core of a game is nonempty. Games with nonempty core are called *balanced*. For a 3-player superadditive game, these conditions reduce to a single additional condition: $v(1,2) + v(1,3) + v(2,3) \leq 2v(1,2,3)$ or $a + b + c \leq 2$ in the normalized version.[4]

The nucleolus is the allocation that results from the lexicographic minimization of the excesses defined as the difference between what a coalition is worth and what it obtains. It shares with the Shapley value three important properties: *efficiency* (the worth of the grand coalition is exactly allocated), *symmetry* (players who contribute equally are allocated identical amounts) and *null player* (a player that does not contribute gets a zero amount). The Shapley value does not necessarily define a core allocation for games whose core is nonempty while the nucleolus selects a core allocation. Dehez and Pacini (2024) show that $SV(N, \bar{v}) \in C(N, \bar{v})$ *if and only if* $a + b \leq 4(1 - c)$.

The core, the Shapley value and the nucleolus are *covariant solution* concepts. Consequently, when dealing with normalized games, the inverse transformation can be applied to the resulting allocations to generate the allocations within the original game:

$$x \in I(N, \bar{v}) \Leftrightarrow y \in I(N, v) \text{ where } y_i = x_i \left(v(N) - \sum_{i \in N} v(i) \right) + v(i). \tag{3}$$

Computing the nucleolus is not straightforward. There are algorithms (see Kohlberg 1971) and software packages.[5] Legros (1981) provides a formula for 3-player games with five cases that correspond to five subsets which form a partition (up to the boundaries) of the parameter space:

$$\begin{aligned} P_1 &= \{(a, b, c) \in P \mid c \leq 1/3\}, \\ P_2 &= \{(a, b, c) \in P \mid 1/3 \leq c \leq 1 - 2b\}, \\ P_3 &= \{(a, b, c) \in P \mid 1 - 2b \leq c \leq 1 - 2a\}, \\ P_4 &= \{(a, b, c) \in P \mid c \geq 1 - 2a \text{ and } c \geq 2(a + b) - 1\}, \\ P_5 &= \{(a, b, c) \in P \mid c \leq 2(a + b) - 1\}. \end{aligned} \tag{4}$$

In P_5, the inequality $c \geq 1 - 2a$ does not appear because it is redundant. In P_1 and in P_2, the inequality $a + b + c \leq 1$ holds and, consequently, all games in these two regions are positive and thereby convex. As already pointed out, all games are convex whenever $c \leq 0.5$ and their Shapley value belongs to the core: the condition $a + b \leq 4(1 - c)$ is satisfied. Games in P_1 and P_4 have a nonempty core while games in P_5 are unbalanced. In the case where $c = 1$, $P = P_4 \cup P_5$, the other regions correspond to boundaries. The following figures illustrate the partition in the (a, b)-plane for two values of c, referring to the regions defined in (4).

In Figure 1, $c = 0.4$. P_1 is empty as it is defined for $c \leq 1/3$. P_2-P_5 form a partition of the set of feasible games. Since $b + c \leq 1$, all games are convex and thereby balanced. They are positive below the dashed line $a + b = 1 - c = 0.6$. The Shapley value belongs to the core for all games since the relation $a + b \leq 4(1 - c)$ is always satisfied.

In Figure 2, $c = 0.8$. P_1 is empty as in the previous case. Now games are convex below the horizontal dashed line $b = 1 - c = 0.2$ and are positive below the dashed line $a + b = 1 - c = 0.2$.

[4] Dehez and Pacini (2024, 2025) show that two third of the 3-player convex games are positive and one fifth of the balanced games are positive.

[5] The website www.tuglabweb.uvigo.es by M.Á. Mirás Calvo and E. Sánchez Rodríguez (University of Vigo), the MATLAB and Mathematica packages by H. Meinhardt (Karlsruhe Institute of Technology).

Balanced games are located below the dashed line $a + b = 2 - c = 1.2$. Games whose Shapley value belongs to the core are located below the dashed line $a + b = 4(1 - c) = 0.8$.

In the particular case where $c = {}^{1}\!/_{3}$, P_1 and P_2 coincide and are equal to the set of admissible games given by the triangle with vertices $(0, 0) - (0, {}^{1}\!/_{3}) - ({}^{1}\!/_{3}, {}^{1}\!/_{3})$. P_3 is the upper boundary of this triangle and the other two regions collapse to the point $({}^{1}\!/_{3}, {}^{1}\!/_{3}, {}^{1}\!/_{3})$. When $c < {}^{1}\!/_{3}$, only region P_1 survives and $P_k = \emptyset$, $k = 2, \ldots, 5$.

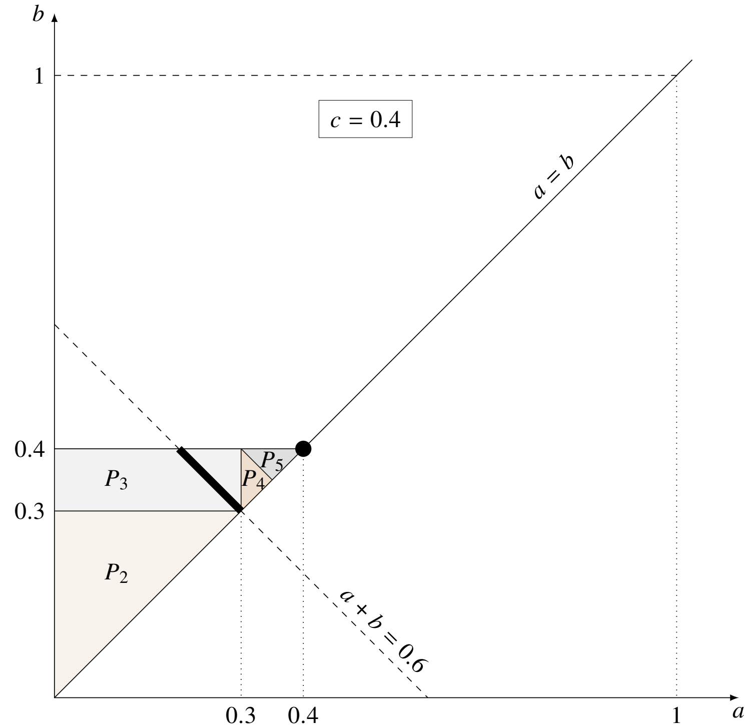

Figure 1

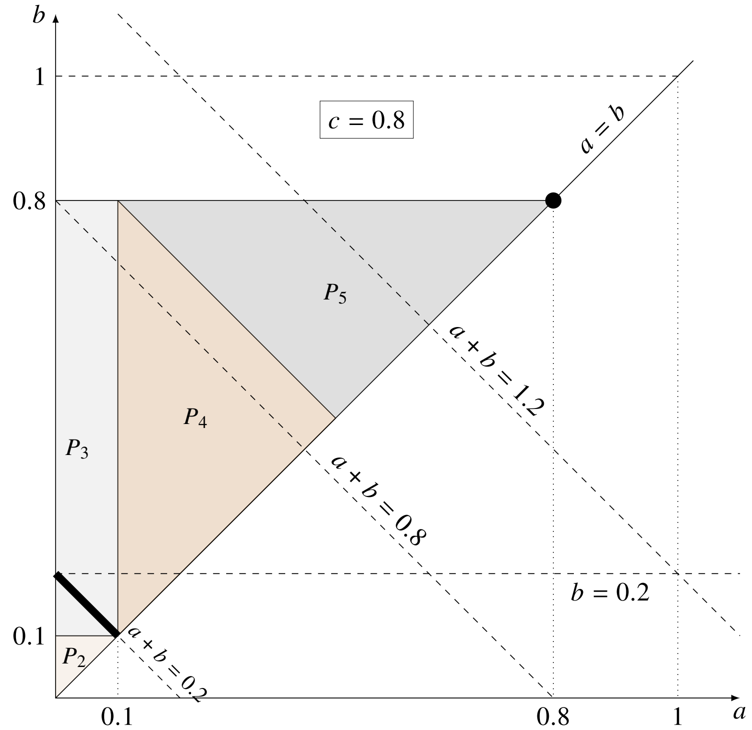

Figure 2

Following Legros (1981), the nucleolus of a 3-player normalized game is given by the following Table. It is easily verified that the nucleolus is continuous across regions.

	x_1	x_2	x_3
P_1	$1/3$	$1/3$	$1/3$
P_2	$\frac{1-c}{2}$	$\frac{1+c}{4}$	$\frac{1+c}{4}$
P_3	$\frac{1-c}{2}$	$\frac{1-b}{2}$	$\frac{b+c}{2}$
P_4	$\frac{1-c}{2}$	$\frac{1+c-2(b-a)}{4}$	$\frac{1+c+2(b-a)}{4}$
P_5	$\frac{1+a+b-2c}{3}$	$\frac{1+a+c-2b}{3}$	$\frac{1+b+c-2a}{3}$

4. Coincidence between the Shapley value and the nucleolus

The following proposition confirms and refines the results of Yokote et al. (2017), with a simple proof.

Proposition 1. *Given the inequalities that characterize the different cases, the Shapley value and the nucleolus of a 3-player normalized game $(N, \bar{v})$ coincide* if and only if*:*

$$\begin{array}{ll} a = b = c & \text{in } P_1 \text{ and } P_5 \\ a + b + c = 1 & \text{in } P_2, P_3 \text{ and } P_4. \end{array}$$

Proof. The different cases are considered separately, referring to (2) for the Shapley value and to the Table for the nucleolus.

P_1: The equations $SV_1(N, \bar{v}) = SV_2(N, \bar{v}) = SV_3(N, \bar{v}) = 1/3$ imply $a = b = c$. If $a = b = c$, the game is symmetric and the Shapley value coincides with the equal division.

P_2: The equation $NUC_1(N, \bar{v}) = SV_1(N, \bar{v})$ reads: $\frac{a+b}{2} + \frac{1-a-b-c}{3} = \frac{1-c}{2}$. It is equivalent to $a + b + c = 1$ which, together with the condition $c \leq 1 - 2b$ that characterizes case 2, implies $a = b$. Consequently, $NUC_i(N, \bar{v}) = SV_i(N, \bar{v})$ for $i = 2, 3$.

P_3: Like in case 2, the equation $NUC_1(N, \bar{v}) = SV_1(N, \bar{v})$ is equivalent to $a + b + c = 1$. The two other equations $NUC_i(N, \bar{v}) = SV_i(N, \bar{v})$ for $i = 2, 3$ are equivalent to $a + b + c = 1$.

P_4: Like in case 2, the equation $NUC_1(N, \bar{v}) = SV_1(N, \bar{v})$ is equivalent to $a + b + c = 1$ which, together with the condition $c \geq 1 - 2a$ that characterizes case 4, implies $a = b$. Consequently, $NUC_i(N, \bar{v}) = SV_i(N, \bar{v})$ for $i = 2, 3$.

P_5: The equation $NUC(N, \bar{v}) = SV(N, \bar{v})$ can be written as:

$$\begin{array}{lcl} \dfrac{1 + a + b - 2c}{3} = \dfrac{a + b}{6} + \dfrac{1 - c}{3} & \Leftrightarrow & a + b = 2c \\ \dfrac{1 + a + c - 2b}{3} = \dfrac{a + c}{6} + \dfrac{1 - b}{3} & \Leftrightarrow & a + c = 2b \\ \dfrac{1 + b + c - 2a}{3} = \dfrac{b + c}{6} + \dfrac{1 - a}{3} & \Leftrightarrow & b + c = 2a \end{array}$$

They are simultaneously satisfied *if and only if* $a = b = c$. □

Referring to the figures, the subsets of values of a and b for which coincidence occurs are the thick line in P_3 and the heavy dot in P_5: coincidence occurs on a subset of measure zero. This is no surprise. As previously noted, the Shapley value and the nucleolus both satisfy the three axioms of efficiency, symmetry, and the null player. *Marginalism* is the fourth axiom that together with the first three, defines the Shapley value. It retains the rules that allocate to each player an amount that only depends on his marginal contributions, independently of the way the other players contribute. This independence axiom is not verified by the nucleolus that is concerned with minimizing the maximum dissatisfaction among coalitions.

Remark 1. The condition $a + b + c = 1$ is satisfied in P_3 and includes the situation where $a = b = c = 1/3$. It is the boundary of the set of positive games, characterized by a dividend of the grand coalition equal to zero.

Remark 2. The condition $a = b = c$ implies symmetry of the normalized game $(N, \bar{v})$.

Remark 3. The core of a 1-convex game (Driessen (1985), Dehez (2024)) is a regular simplex whose center of gravity is the nucleolus. In the case of a game that is both convex and 1-convex, the nucleolus and the Shapley value coincide. Within a 3-player normalized game, only one game fits these conditions, namely the symmetric game defined by $a = b = c = 1/2$.

5. Differential treatment of players in the Shapley value and the nucleolus

Looking at how players are being treated within the two allocation rules, we observe that, in the normalized game, they are ordered in terms of their marginal contributions: $(a, b, 1 - c) \leq (a, c, 1 - b) \leq (b, c, 1 - a)$. The Shapley value allocates to each player an amount that depends exclusively on his marginal contribution. Consequently, what players obtain must follow the same order. The nucleolus, as given by the Table, shares that ordering in all cases. This is immediate in cases 1, 2 and 5. It is easily verified that it is so as well in the other two cases, using the inequalities that qualify them.

How are the players being comparatively treated by the Shapley value and the nucleolus? We first consider the weakest and the strongest players, players 1 and 3.

Proposition 2. *In P_1, P_2 and for the positive games in P_3, player 1 is better treated by the nucleolus while player 3 is better treated by the Shapley value. The reverse applies in P_4, P_5 and for the games in P_3 that are not positive.*

Proof. We refer to the nucleolus defined by the Table and to the Shapley value defined by (2). Let us define $\Delta_i = SV_i(N, \bar{v}) - NUC_i(N, \bar{v})$. The sign of Δ_1 is given by the sign of $a + b - 2c$ in P_1 and by the sign of $a + b + c - 1$ in P_2. Because $a \leq b \leq c \leq 1$ and $a + b + c \leq 1$, both are non-positive: the nucleolus treats better the first player in P_1 and P_2.

The sign of Δ_3 is given by the sign of $b + c - 2a$ in P_1 and by the sign of $1 + 2b - 4a - c$ in P_2. Because $a \leq b \leq c \leq 1$ and $c \leq 1 - 2b$, both are non-negative: the Shapley value treats better the last player in P_1 and P_2.

The sign of Δ_1 is given by the sign of $a + b + c - 1$ in P_3. Then the first player is better treated by the nucleolus when the game is positive ($a + b + c \leq 1$), by the Shapley value otherwise.

The sign of Δ_3 is given by the sign of $1 - a - b - c$ in P_3; the third player is better treated by the Shapley value when the game is positive and by the nucleolus otherwise.

The sign of Δ_1 is given by the sign of $a + b + c - 1$ in P_4 and by the sign of $2c - a - b$ in P_5. Because $a \leq b \leq c \leq 1$ and $a + b + c \geq 1$, both are non-negative: the Shapley value treats better the first player in P_4 and P_5.

The sign of Δ_3 is given by the sign of $1 + 2a - 4b - c$ in P_4 and by the sign of $2a - b - c$ in P_5. Because $a \leq b \leq c \leq 1$ and $c \geq 1 - 2a$ both are non-positive: the nucleolus treats better the last player in P_4 and P_5. □

Let us now consider the middle player.

Proposition 3. *Player 2 is better treated by the Shapley value in P_1 if $2b < a + c$ and by the nucleolus if the reverse inequality holds. In P_5, the argument is reversed: player 2 is better treated by the nucleolus if $2b < a + c$. In P_3, the treatment of player 2 coincides with the treatment of player 1.*

Proof. The sign of Δ_2 is given by the sign of $a + c - 2b$ in P_1 and by the sign of $2b - a - c$ in P_5. The sign of Δ_2 is given by the sign of $a + b + c - 1$ in P_3. □

Nothing precise can be said about the treatment of player 2 in the subsets P_2 and P_4. Simulations reveal that in P_2, the second player is favored by the Shapley value approximately $1/2$ of the cases. In P_4, he is favored by the Shapley value in about $2/3$ of the cases.

6. Applications

We illustrate our results with two classes of games, airport and bidder-collusion games, for which coincidence has been studied by Yokote et al. (2017). We show that coincidence never occurs in proper airport games while it occurs in bidder games when the first two players have the same evaluations. We also show how the two solutions differently treat players.

6.1. Airport games

Airport games form a class of cost-sharing games introduced by Littlechild and Owen (1973), for which Littlechild (1974) and Sönmez (1993) give recursive formulas for the nucleolus. The airport game associated to the cost parameters $0 < c_1 \leq c_2 \leq \ldots \leq c_n$ is given by $c(S) = max_{i \in S}\, c_i$. In what follows, we will consider the associated surplus-sharing game (N, v) defined by $v(S) = \Sigma_{i \in S} c_i - c(S)$. The 3-player normalized game $(N, \bar{v})$ is then defined by the following parameters:

$$a = b = \frac{c_1}{c_1 + c_2} \quad \text{and} \quad c = \frac{c_2}{c_1 + c_2}.$$

The parameters (a, b, c) belong to P. They satisfy the inequalities $1 < a + b + c \leq 3/2 < 2$ and $b + c = 1$, confirming that the surplus-sharing game is convex and thereby balanced. It is however not positive. Moreover, $c \geq 1/2$. The inequality $c \geq 1 - 2a$ being satisfied, we fall in P_4 if $2c_2 \geq 3c_1$ or in P_5 if the reverse inequality holds. In P_4, using (3), the nucleolus of the normalized and surplus-sharing games are given by:

$$NUC_1(N, \bar{v}) = \frac{c_1}{2(c_1 + c_2)} \quad \longrightarrow \quad NUC_1(N, v) = \frac{c_1}{2}$$

$$NUC_2(N, \bar{v}) = NUC_3(N, \bar{v}) = \frac{c_1 + 2c_2}{4(c_1 + c_2)} \quad \longrightarrow \quad NUC_2(N, v) = NUC_3(N, v) = \frac{c_1}{4} + \frac{c_2}{2}$$

In P_5, they are given by:

$$NUC_1(N, \bar{v}) = \frac{3c_1 - c_2}{3(c_1 + c_2)} \quad \longrightarrow \quad NUC_1(N, v) = c_1 - \frac{c_2}{3}$$

$$NUC_2(N, \bar{v}) = NUC_3(N, \bar{v}) = \frac{2c_2}{3(c_1 + c_2)} \quad \longrightarrow \quad NUC_2(N, v) = NUC_3(N, v) = \frac{2c_2}{3}$$

Following (2), the Shapley value is given by $SV(N, v) = \left(\frac{2c_1}{3}, \frac{c_1}{6} + \frac{c_2}{2}, \frac{c_1}{6} + \frac{c_2}{2}\right)$.

By convexity, it defines a core allocation. Proposition 1 is confirmed. In the subset P_4, given that $c_1 > 0$, coincidence between the Shapley value and the nucleolus never occurs simply because $a + b + c > 1$. In the subset P_5, coincidence occurs *if and only if* $c_1 = c_2$ and both solutions coincide with equal division. In that case, the game is equivalent to a 2-player airport game. Referring to Proposition 2 and Proposition 3, we observe that in both regions the first player prefers the allocation proposed by the Shapley value while the other two players prefer the allocation proposed by the nucleolus.

6.2. Bidder collusion game

English auctions are ascending auctions. Graham et al. (1990) have considered a situation where bidders' valuations are common knowledge and given by $0 < b_1 \leq b_2 \leq ... \leq b_n$. If a coalition S forms, it behaves as a single bidder: in the non-cooperative game opposing S to $N \setminus S$, remaining active until the bidding reaches $max_{i \in S} b_i$ is the dominant strategy for S and remaining active until the bidding reaches $max_{i \in N \setminus S} b_i$ is the dominant strategy for $N \setminus S$. The resulting payoff for coalition S is then given by:

$$v(S) = \begin{cases} b_n & \text{if } S = N, \\ b_n - max_{i \notin S}\, b_i & \text{if } S \neq N. \end{cases}$$

In the case of three players, the game is given by $v = (0, 0, b_3 - b_2\, |0, b_3 - b_2, b_3 - b_1|\, b_3)$. The associated normalized game is then defined by $a = b = 0$ and $c = 1 - b_1/b_2$. The inequality $a + b + c < 1$ being satisfied, the game is positive and thereby convex: only the subsets P_1 and P_2 are concerned, depending on the ratio b_1/b_2. If $b_1/b_2 \geq 2/3$, the nucleolus is given by the equal division or, in terms of the original game, using (3):

$$NUC_1(N, v) = NUC_2(N, v) = \frac{b_2}{3},$$
$$NUC_3(N, v) = b_3 - \frac{2}{3} b_2.$$

If instead $b_1/b_2 \leq 2/3$, they are given by:

$$\begin{aligned} NUC_1(N, \bar{v}) &= \frac{b_1}{2b_2} \\ NUC_2(N, \bar{v}) = NUC_3(N, \bar{v}) &= \frac{1}{4}\left(2 - \frac{b_1}{b_2}\right) \end{aligned} \quad \rightarrow \quad \begin{aligned} NUC_1(N, v) &= \frac{b_1}{2}, \\ NUC_2(N, v) &= \frac{b_2}{2} - \frac{b_1}{4}, \\ NUC_3(N, v) &= b_3 - \frac{b_1}{4} - \frac{b_2}{2}. \end{aligned}$$

By (2), the Shapley value is given by $SV(N, v) = \left(\frac{b_1}{3}, \frac{b_2}{2} - \frac{b_1}{6}, b_3 - \frac{b_2}{2} - \frac{b_1}{6}\right)$.

We observe that coincidence occurs in P_1 if and only if $b_1 = b_2$ while it never occurs in P_2, except if $b_1 = 0$, in which case we are reduced to a 2-player game. Assuming $b_2 > b_1 > 0$ and referring to Proposition 2 and Proposition 3, we observe that, in both cases, the nucleolus treats better the first player while the Shapley value treats better the other two players.

7. Concluding remarks

By limiting ourselves to the case of superadditive essential games with three players and applying the formula of Legros (1981), we propose a simple framework where the coincidence between the Shapley value and the nucleolus is confirmed to be a fragile property. Coincidence only occurs under specific configurations of parameters that reflect symmetry or a particular form of positivity of the normalized game, in line with the results of Yokote et al. (2017). Beyond these exceptional cases, the two solutions result in different allocations. We show how players rank the two solutions depending on the parameters defining the game. The nucleolus favors the position of the weakest player in P_1, P_2, and the positive games of P_3, while the Shapley value favors the strongest player. The reverse occurs in P_4, P_5, and the non-positive games of P_3. Overall, the share of games in which the first player benefits more from the Shapley value than from the nucleolus is significant, around 83% of the cases in P. As for the second player, his advantage shifts between the two solutions depending on the parameter configuration.

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