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A note on the relation between the Shapley value and the core of 3-player transferable utility games

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Abstract

We reconsider the necessary and sufficient conditions under which the Shapley value of a 3-player superadditive game belongs to the core. We then compute the proportion of games whose Shapley value belongs to the core within the set of balanced superadditive games.

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1 Introduction

The specification of necessary and sufficient conditions under which the Shapley value belongs to the core of a transferable utility game is an old question: under which conditions does the Shapley value define an allocation that is stable, in the sense that it raises no objection from individual players or coalitions? This question has been dealt with in a number of papers: Iñarra and Usategui (1993), Izawa and Takahashi (1998), Yokote et al. (2015, 2017) and Abe and Nakada (2022).

Our objective is more modest: obtain a simple and interpretable condition that is necessary and sufficient to ensure that the Shapley value belongs to the core of 3-player superadditive games. We use this result to compute the proportion of superadditive and balanced games whose Shapley value belongs to the core. It is found to be equal to 0.64, in line with a previous conjecture by Fukiharu (2013) based on numerical simulations. This confirms that strong restrictions must be imposed on balancedness. On the other hand it is more than twice the share of the set of convex games, showing that the Shapley value defines a core allocation for a considerably larger superset of the set of convex games.

The paper is organized as follows. Section 2 introduces the concept of transferable utility game. The core and the Shapley value are then defined in the following two sections. In Section 5, working on normalized 3-player games, it is shown that the 2-player coalition with the highest worth is pivotal in determining whether or not the Shapley value belongs to the core. This property is then used to specify a necessary and sufficient condition on the distribution of the worth of 2-player coalitions under which the Shapley value belongs to the core. The proportions of the latter set of games and of the set of convex games within the wider set of superadditive and balanced games are then dealt with in Section 6. Concluding remarks are offered in the last section.

2 Transferable utility games

A *transferable utility game* on a set of players $N = \{1, \ldots, n\}$ is defined by a function v that associates a real number to every subset $S \subset N$. Here, $v(S)$ is the "worth" of coalition S expressed in terms of some commodity-money.

A game (N, v) is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for all pairs of disjoint coalitions S and T. Superadditivity is a natural assumption: the players have a potential interest in forming the "grand coalition" N and the question concerns the division of $v(N)$ among the n players. Under superadditivity, we have $\sum_{i \in N} v(i) \leq v(N)$. A game is *essential* if the latest inequality is strict: it is possible to allocate $v(N)$ so as to give to each player more than their individual worth.

A game (N, v) is *convex* if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for arbitrary pairs of coalitions S and T; convexity implies superadditivity. Convexity can be defined in terms of marginal contributions. The *marginal contribution* of player i to coalition S is defined by $MC_i^v(S) = v(S) - v(S \backslash i)$. A game is convex *if and only if* marginal contributions are non-decreasing with respect to coalition sizes: $i \in S \subset T \Rightarrow MC_i^v(S) \leq MC_i^v(T)$.[1]

Without loss of generality, essential games can be normalized such that individual

[1] A result due to Shapley (1971) and Ichiishi (1981). More on TU-games in Maschler et al. (2020) and Dehez (2024).

worth's equal zero and the worth of the grand coalition equals 1:

$$\bar{v}(S) = \frac{v(S) - \sum_{j \in S} v(j)}{v(N) - \sum_{j \in N} v(j)} \quad \text{for all } S \subset N.$$

A game and its normalized version are *strategically equivalent*: $\bar{v}(S) = \alpha\, v(S) + \beta(S)$ where

$$\alpha = \frac{1}{v(N) - \sum_{j \in N} v(j)} \quad \text{and} \quad \beta_i = \frac{-v(i)}{v(N) - \sum_{j \in N} v(j)} \qquad (i = 1, \ldots, n).$$

The allocation $(x_1, \ldots, x_n)$ associated to the game (N, v) and the allocation $(\bar{x}_1, \ldots, \bar{x}_n)$ associated to its normalized version $(N, \bar{v})$ are related by the equations $\bar{x}_i = \alpha\, x_i + \beta_i$, $(i = 1, \ldots, n)$. This is the property of *covariance* that will be used later.

Notation. Finite sets are denoted by upper-case letters and lower-case letters are used to denote their cardinals: $t = |T|$, $s = |S|$, ... For a given vector x, $x(S)$ denotes the sum of its coordinates over the set S. Braces are sometimes omitted for coalitions: for instance, $v(i, j)$ replaces $v(\{i, j\})$.

We denote by Γ^3 the set of all 3-player superadditive and essential games. Given a game $(N, v) \in \Gamma^3$ and following Tchantcho et al. (2012), we define successively:

$$a(\bar{v}) = \min\left(\bar{v}(1,2), \bar{v}(1,3), \bar{v}(2,3)\right),$$
$$c(\bar{v}) = \max\left(\bar{v}(1,2), \bar{v}(1,3), \bar{v}(2,3)\right).$$

Players are labelled in such a way that $a(\bar{v}) = \bar{v}(1,2)$ and $c(\bar{v}) = \bar{v}(2,3)$. The worth of the intermediary coalition is then denoted by $b(\bar{v}) = \bar{v}(1,3)$. Superadditivity of the original game implies that the three parameters $a(\bar{v})$, $b(\bar{v})$ and $c(\bar{v})$ belong to the interval $[0, 1]$. Hence, we will consider normalized games defined by vectors $(0, 0, 0|a, b, c|1)$ where $0 \leq a \leq b \leq c \leq 1$.

Superadditivity and convexity are preserved through normalization. In particular, a game is convex *if and only if* its normalized version $(N, \bar{v})$ is convex. The following proposition characterizes convexity in terms of the parameters $b(\bar{v})$ and $c(\bar{v})$.

Proposition 1. *A game $(N, v) \in \Gamma^3$ is convex if and only if $b(\bar{v}) + c(\bar{v}) \leq 1$.*

Proof. For the coalitions $S = \{1, 3\}$ and $T = \{2, 3\}$, we have $\bar{v}(S) + \bar{v}(T) = b(\bar{v}) + c(\bar{v})$ and $\bar{v}(S \cup T) + \bar{v}(S \cap T) = 1$. Hence, if the game is convex, the inequality holds. Inversely, assume that the inequality holds. Clearly the convexity condition holds independently from the inequality $b(\bar{v}) + c(\bar{v}) \leq 1$ in either one of the following two cases: given two distinct coalitions S and T in N, (i) either S or T is the grand coalition or (ii) S and T have empty intersection. There remain two cases to consider. If $S = \{i, j\}$ and $T = \{i\}$, we have $\bar{v}(S \cup T) + \bar{v}(S \cap T) = \bar{v}(S)$ and $\bar{v}(S) + \bar{v}(T) = \bar{v}(S)$. If $S = \{i, j\}$ and $T = \{i, k\}$, we have $\bar{v}(S \cup T) + \bar{v}(S \cap T) = 1$ and $\bar{v}(S) + \bar{v}(T) \leq b(\bar{v}) + c(\bar{v})$. It is the only case where the inequality $b(\bar{v}) + c(\bar{v}) \leq 1$ is being used. □

3 The core

Two minimal conditions are to be imposed on allocations. *Efficiency* requires that the entire value of a game is distributed to the players. *Individual rationality* requires that all players should obtain at least their individual worth. Together, these two requirements define *imputations* and superadditivity ensures their existence[2]. The *core* is the set of

[2]The imputation set of a normalized n-player game is the unit simplex of dimension $n - 1$.

allocations obtained by extending to coalitions the requirement of individual rationality[3]. In general, given a game (N, v) the core is the set of allocations that give to each coalition at least its worth:

$$C(N, v) = \{x \in \mathbb{R}^n \,| x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } S \subset N\}.$$

Core allocations are stable: no coalition is in a position to raise an objection against a core allocation. The core may be empty. Bondareva (1963) and Shapley (1967) have independently given *necessary and sufficient* conditions under which the core of a game is nonempty. This is the notion of *balancedness*. A 3-player game (N, v) is balanced *if and only if* the following five inequalities are verified:

$$\begin{aligned} v(1) + v(2) + v(3) &\leq v(1, 2, 3), \\ v(1, 2) + v(3) &\leq v(1, 2, 3), \\ v(1, 3) + v(2) &\leq v(1, 2, 3), \\ v(2, 3) + v(1) &\leq v(1, 2, 3), \\ v(1, 2) + v(1, 3) + v(2, 3) &\leq 2v(1, 2, 3). \end{aligned} \tag{1}$$

For superadditive games, only the last inequality matters[4]. The core of the normalized version of a game $(N, v) \in \Gamma^3$ is defined by:

$$C(N, \bar{v}) = \{x \in \mathbb{R}^n \,| x_1 + x_2 + x_3 = 1,\; 0 \leq x_i \leq 1 - \bar{v}(j, k) \text{ for all } i \neq j \neq k\}.$$

and the condition of balancedness is given by:

$$a(\bar{v}) + b(\bar{v}) + c(\bar{v}) \leq 2. \tag{2}$$

Remark 1. *Because the way the three terms in* (2) *are ordered, balancedness of a 3-player normalized game implies that* $a(\bar{v}) \leq \frac{2}{3}$.

4 The Shapley value

Looking at allocation rules, the simplest one is equal division. Equal division of the surplus is slightly more sophisticated. It is given by:

$$EDS_i(N, V) = v(i) + \frac{1}{n}\left(v(N) - \sum\nolimits_{i \in N} v(i)\right) \quad (i = 1, \ldots, n). \tag{3}$$

It does not take into account differences in players' contribution. The *Shapley value* is an allocation rule introduced by Shapley in 1953. It is uniquely defined by a set of axioms that includes *equal treatment of equals*. Several formulas can be used to compute the Shapley value. We take the original one where players get a weighted average of their marginal contributions:

$$SV_i(N, v) = \sum_{S \subset N} \alpha_n(s) MC_i^v(S) \qquad (i = 1, \ldots, n)$$

where $\alpha_n(s) = \frac{(s-1)!(n-s)!}{n!}$. Applied to the normalized version of a 3-player game, we have:

$$SV_i(N, \bar{v}) = \frac{1}{6}\left(\bar{v}(i, j) + \bar{v}(i, k)\right) + \frac{1}{3}\left(1 - \bar{v}(j, k)\right).$$

[3]The notion of core was introduced by Gillies (1953) in relation to the concept of von Neumann-Morgenstern stable set and was established as a solution concept by Shapley (1955). See Zhao (2018).

[4]A game can be balanced but not superadditive, like for instance the game $(1, 2, 3|2, 5, 8|10)$.

Hence, dropping the dependence upon $\bar{v}$, we have:

$$
\begin{aligned}
SV_1(N,\bar{v}) &= \frac{1}{6}(a+b) + \frac{1}{3}(1-c)\,,\\
SV_2(N,\bar{v}) &= \frac{1}{6}(a+c) + \frac{1}{3}(1-b)\,,\\
SV_3(N,\bar{v}) &= \frac{1}{6}(b+c) + \frac{1}{3}(1-a)\,.
\end{aligned} \tag{4}
$$

5 Relations between the core and the Shapley value

In general, nothing ensures that the Shapley value belongs to the core of a balanced game. Shapley (1971) has proven that it is the case for convex games, but it may also be the case for nonconvex games. Here are two possible cases.

Remark 2. *A superadditive and inessential game is additive, in which case, the core consists of the allocation that assigns to players their individual worth and it coincides with the Shapley value*[5]*. This justifies our attention to essential games.*

Remark 3. *In the 2-player case, the core is nonempty if and only if superadditivity holds, in which case the Shapley value coincides with the equal division of the surplus* (3) *and belongs to the core.*

The core and the Shapley value are covariant solutions. Hence, the Shapley value of a game $(N,v) \in \Gamma^3$ belongs to its core *if and only if* the Shapley value of its normalized version $(N,\bar{v})$ belongs to its core. The Shapley value of the normalized version (4) defines an imputation: the SV_i's are non-negative and sum up to one. It defines a core allocation *if and only if* $SV_i(N,\bar{v}) + SV_j(N,\bar{v}) \geq \bar{v}(i,j)$ for all pairs (i,j):

$$
\begin{aligned}
\frac{a+b}{6} + \frac{1-c}{3} + \frac{a+c}{6} + \frac{1-b}{3} \geq a &\Leftrightarrow \frac{2(1-a-b-c)}{3} + \frac{b+c}{2} \geq 0,\\
\frac{a+b}{6} + \frac{1-c}{3} + \frac{b+c}{6} + \frac{1-a}{3} \geq b &\Leftrightarrow \frac{2(1-a-b-c)}{3} + \frac{a+c}{2} \geq 0,\\
\frac{a+c}{6} + \frac{1-b}{3} + \frac{b+c}{6} + \frac{1-a}{3} \geq c &\Leftrightarrow \frac{2(1-a-b-c)}{3} + \frac{a+b}{2} \geq 0.
\end{aligned} \tag{5}
$$

These three inequalities appear in Yokote et al. (2015). The following proposition says that the coalition with the highest value is critical in determining the stability of the Shapley value.

Proposition 2. *The Shapley value of game $(N,v) \in \Gamma^3$ belongs to the core if and only if, within its normalized version $(N,\bar{v})$, the 2-player coalition with the highest worth raises no objection.*

Proof. Given (5) and assuming that the inequalities $0 \leq a \leq b \leq c$ hold, $SV_2 + SV_3 \geq c$ implies $SV_1 + SV_3 \geq b$ and $SV_1 + SV_2 \geq a$. □

The following proposition is an immediate corollary of Proposition 2.

Proposition 3. *Given a normalized game $(0,0,0|a,b,c|1)$ where $0 \leq a \leq b \leq c$, the Shapley value belongs to its core if and only if:*

$$
c \leq 1 - \frac{a+b}{4}. \tag{6}
$$

[5] A game is additive if the worth of any coalition equals the sum of its members' worth.

Example. Consider the following 3-player game $(25, 15, 20|40, 69, 66|z)$. The balancedness conditions (1) are satisfied for $z \geq 91$. Under this inequality, the game is superadditive and essential. Its normalized version is given by

$$a = \bar{v}(1,2) = 0, \quad b = \bar{v}(1,3) = \frac{24}{z-60} \quad \text{and} \quad c = \bar{v}(2,3) = \frac{31}{z-60}.$$

Convexity applies for $z \geq 115$. Condition (6) is satisfied with equality for $z = 97$: the Shapley value of the original game (N, v) belongs to the core *if and only if* $z \geq 97$. For $z = 97$, it is given by the allocation $(31, 24.5, 41.5)$ that lies on the boundary of the core.

Remark 4. *The inequality $c \leq \frac{2}{3}$ is a sufficient condition for the Shapley value to be stable.*

Indeed, (6) can be written as $a + b + c \leq 4 - 3c$, an inequality that is verified, for a balanced game, if $c \leq \frac{2}{3}$. In the case of a symmetric game, where the worth of a coalition only depends on the its cardinality, (6) reduces to $k \leq \frac{2}{3}$ where k is the worth of a 2-player coalition.

For a general 3-player superadditive game (N, v), the above results translate as follows:

Proposition 4. *Given a game $(N, v) \in \Gamma^3$, the Shapley value belongs to the core if and only if the following inequalities hold for every player $i \in N$:*

$$MC_i^v(i) + \frac{MC_i^v(i,j) + MC_i^v(i,k)}{2} \leq 2MC_i^v(N). \tag{7}$$

Proof. Relabel the players in N in such a way that

$$v(1,2) - v(1) - v(2) \leq v(1,3) - v(1) - v(3) \leq v(2,3) - v(2) - v(3).$$

Dividing the above inequalities by $\Delta = v(N) - v(1) - v(2) - v(3)$, we get $\bar{v}(1,2) \leq \bar{v}(1,3) \leq \bar{v}(2,3)$. These numbers belong to the interval $[0, 1]$ by superadditivity, so that Proposition 3 applies, where (6) becomes $4(1 - \bar{v}(2,3)) \geq \bar{v}(1,2) + \bar{v}(1,3)$. In view of Proposition 3 and strategic equivalence, the latter condition is necessary and sufficient for the Shapley value to belong to the core. And, in view of Proposition 2, it is also necessary and sufficient for the three inequalities $4(1 - \bar{v}(j,k)) \geq \bar{v}(i,j) + \bar{v}(i,k)$ to be satisfied for all $i, j, k \in \{1, 2, 3\}$, $i \neq j \neq k$. Multiplying both sides of the last inequalities by Δ, they become

$$4(\Delta - v(j,k) + v(j) + v(k)) \geq v(i,j) + v(i,k) - 2v(i) - v(j) - v(k)$$

or equivalently

$$4MC_i^v(N) \geq 2MC_i^v(i) + MC_i^v(i,j) + MC_i^v(i,k).$$

Then, dividing by 2, we obtain the inequalities (7). □

Condition (7) says that the average of the average marginal contributions of a player does not exceed his marginal contribution to the grand coalition. Proposition 4 states that, if this condition is met by all players, the Shapley value is in the core. Conversely, it is enough for a player not to satisfy (7) to make the Shapley value unstable since the complementary 2-player coalition could object. In the above example, condition (7) is met by all players for $z \geq 97$. For lower values of z, (7) is not satisfied by (at least) the first player: coalition $\{2, 3\}$ could object since $SV_2 + SV_3 < 66 = v(2,3)$.

6 On the proportion of games whose Shapley value is stable

Using simulations, Fukiharu (2013) has estimated that the proportion of 3-player superadditive and balanced games whose Shapley value belongs to the core lies between 0.60 and 0.65. Using the strategic equivalence between a game and its normalized version, Proposition 3 can be used to compute exactly this proportion for games of the form $(0, 0, 0|a, b, c|1)$ assuming that $0 \le a \le b \le c \le 1$.

Let us fix a particular value of the parameter a, where $a \in \left[0, \frac{2}{3}\right]$ as a consequence of Remark 1. Working in the space of coordinates b and c, we distinguish two subsets of the set of balanced games $B(a) = \{(b, c)|0 \le a \le b \le c \le 1 \text{ and } b + c \le 2 - a\}$: the set $SC(a)$ of games whose Shapley value is stable and the set $Con(a)$ of convex games. We know that $Con(a) \subset SC(a) \subset B(a)$ for all $a \in \left[0, \frac{2}{3}\right]$.

We distinguish two cases: Figure 1 corresponds to a situation where a is below $\frac{1}{2}$ while Figure 2 corresponds to a situation where a is above $\frac{1}{2}$.

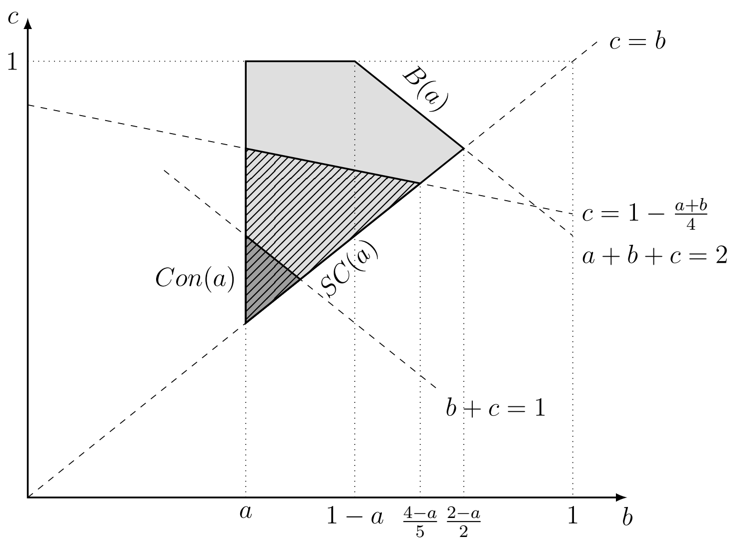

Figure 1: $a < \frac{1}{2}$

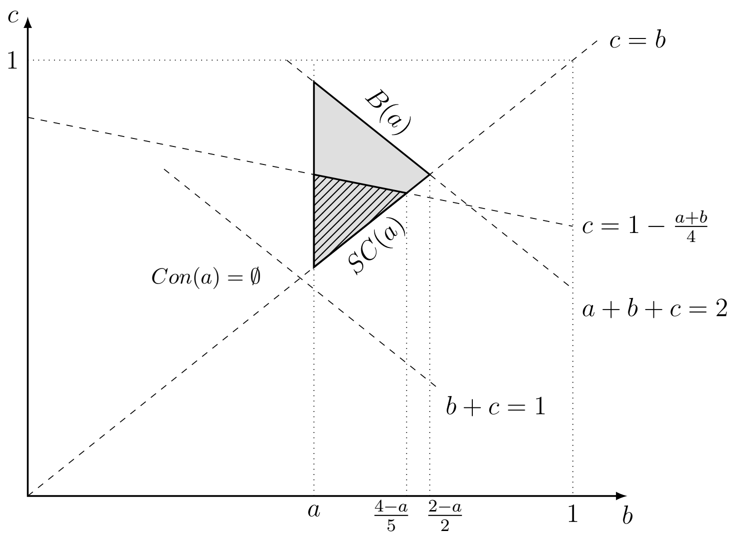

Figure 2: $a > \frac{1}{2}$

In both figures, the set $B(a)$ is the total shaded area. According to Proposition 1, the subset $Con(a)$ is nonempty whenever $b+c \leq 1$. We observe that $Con(a) = \emptyset$ for a larger than $\frac{1}{2}$. Indeed, $a > \frac{1}{2}$ implies $b+c > 1$. The set $SC(a)$ contains the pairs $b+c$ such that $c \leq 1 - \frac{a+b}{4}$. It is nonempty for all $a \in \left[0, \frac{2}{3}\right]$.

We have all elements to compute the shares of Shapley-stable and convex games within the set of 3-player superadditive and balanced games, assuming that the values of the parameters are equally likely. For a given value of $a \in \left[0, \frac{2}{3}\right]$ and referring to the above figures, we have:

$$B(a) = \begin{cases} \int_a^{1-a} (1-b)db + \int_{1-a}^{1-\frac{a}{2}} (2-a-2b)db = \frac{1}{2} - a + \frac{a^2}{4} & \text{for } a \in \left[0, \frac{1}{2}\right] \\ \int_a^{1-\frac{a}{2}} (2-a-2b)db = \frac{1}{4}(2-3a)^2 & \text{for } a \in \left[\frac{1}{2}, \frac{2}{3}\right] \end{cases},$$

$$SC(a) = \int_a^{\frac{4-a}{5}} \left(1 - \frac{a+b}{4} - b\right) db = \frac{(2-3a)^2}{10} \qquad \text{for } a \in \left[0, \frac{2}{3}\right],$$

$$Con(a) = \int_a^{\frac{1}{2}} (1-2b)\, db = a^2 - a + \frac{1}{4} \qquad \text{for } a \in \left[0, \frac{1}{2}\right].$$

Hence, we have successively:

$$B = \int_0^{\frac{2}{3}} B(a)da = \frac{5}{36} \approx 0.139,$$

$$SC = \int_0^{\frac{2}{3}} SC(a)da = \frac{4}{45} \approx 0.089 \quad \to \frac{SC}{B} = \frac{4/45}{5/36} = 0.64,$$

$$Con = \int_0^{\frac{1}{2}} Con(a)da = \frac{1}{24} \approx 0.0417 \quad \to \begin{cases} \frac{Con}{B} = \frac{1/24}{5/36} = 0.3, \\ \frac{Con}{SC} = \frac{1/24}{4/45} = 0.46875. \end{cases}$$

Hence, 64% of the superadditive and balanced games are Shapley-stable, confirming Fukiharu estimation, 30% of the superadditive and balanced games are convex and about 47% of the Shapley-stable games are convex. We observe that the stability condition (6) imposes a considerable relaxation of convexity. Finally we add the following:

Remark 5. *Positive games, games whose Harsanyi dividends are non-negative, are convex and form a class on which solution concepts tend to converge*[6]. *For a 3-player normalized game* $(0, 0, 0|a, b, c|1)$*, positivity applies whenever* a*,* b *and* c *are non-negative and* $a + b + c \leq 1$*. Computations reveal that 20% of the convex games are positive.*

7 Concluding remarks

We have only considered 3-player games where we obtain a single inequality involving three parameters. Moving to a larger set of players would require finding inequalities involving a much larger number of parameters. Considering n-player games, Izawa and

[6] See Harsanyi (1959) and Dehez (2017) for an extensive study of Harsanyi dividends and related concepts.

Takahashi (1998) have introduced the following necessary and sufficient conditions

$$\sum_{S \subset N} \sum_{i \in S \cap T} \alpha_n(s) \left((v(S) - v(S \backslash i)) - (v(S \cap T) - v((S \cap T) \backslash i)) \right) \geq 0$$

for all $T \subset N$, where the coefficients $\alpha_n(s)$ are the Shapley weights. Applied to the normalized version of a game in Γ^3, the above condition is equivalent to (6) for $T = \{2, 3\}$.

Yokote, Funaki, and Kamijo (2017) provide a necessary and sufficient condition that applies to 3-player 0-normalized games (Remark 2, p.5). For superadditive normalized games, it reduces to the following two inequalities:

$$4c + b + a \leq 4,$$
$$2c \leq a + b + 2.$$

Since $c \leq 1$, the second inequality is always verified: only the first one matters and is equivalent to (6).

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