Economics Bulletin

Volume 40, Issue 4

Coalitions of concerned voters: a characterization of the majority rule

Adrian Miroiu
National School of Political and Administrative Studies

Abstract

The paper gives a characterization of the simple majority rule by appealing to collections of coalitions of voters called frames. It is proved that consistent, monotonic, anonymous and responsive frames include only coalitions consisting in more than half of the concerned voters. An alternative is selected by the simple majority rule if the coalition of voters who favor it is a member of the frame. The property of responsiveness, a far cry of May's (1952) original axiom, is further discussed and some properties of it are proved.

Citation: Adrian Miroiu, (2020) "Coalitions of concerned voters: a characterization of the majority rule", *Economics Bulletin*, Volume 40, Issue 4, pages 2717-2722
Contact: Adrian Miroiu - adrian.miroiu@snspa.ro.
Submitted: May 11, 2020. **Published:** October 12, 2020.

Economics Bulletin

Submission Number: EB-20-00434

COALITIONS OF CONCERNED VOTERS: A CHARACTERIZATION OF THE MAJORITY RULE

Adrian Miroiu
National School of Political and Administrative Studies

Abstract

The paper gives a characterization of the simple majority rule by appealing to collections of coalitions of voters called frames. It is proved that consistent, monotonic, anonymous and responsive frames include only coalitions consisting in more than half of the concerned voters. An alternative is selected by the simple majority rule if the coalition of voters who favor it is a member of the frame. The property of responsiveness, a far cry of May's (1952) original axiom, is further discussed and some properties of it are proved.

Submitted: May 11, 2020.

1. Introduction

Following the classic result of May (1952), the simple majority rule μ was offered a large number of characterizations. The idea is to identify a group of properties μ and only μ satisfies. These properties describe the way in which the majority choice varies when different configurations of individual attitudes are met. Sometimes the focus is on a single society and varying individual preferences or choices, as in May (1952) and Campbell and Kelly (2000). Other authors allow for varying of both societies and individual attitudes: Asan and Sanver (2002), Woeginger (2005), Alcantud (2020). A third approach, followed by Xu and Zhong (2010) and Quesada (2011), is to keep unchanged individual attitudes, but let vary the society at which the collective selection is done. Wu, Xu and Zhong (2015) characterized approval voting in a similar way.

The present paper is in line with this third approach. However, it works in a somehow different framework. I characterize μ by appealing to the concept of coalition of voters in a group and give necessary and sufficient conditions for identifying winning coalitions. To do this, I define collections of coalitions called frames and show that if they satisfy the properties of consistency, monotonicity, anonymity and responsiveness, all the coalitions they include must contain more than half of the concerned voters. An alternative is selected by μ if the coalition of voters who favor it is a member of the frame. I shall also spend some time to discuss two implications of the property of responsiveness.

2. A characterization of the majority rule

Let G be a finite non-empty set. We call voters and denote by v, v', v'' etc. the elements of G. Subsets A, B, C etc. of G denote coalitions. A^C is the complement of coalition A relative to G, i.e. the set $G - A$. The expression $\#A$ denotes the number of members of A. A collection $\mathbf{F}_G \subseteq \wp(G)$ of coalitions in G is called a frame on G. I shall say that members of $\mathbf{F}_G$ are winning coalitions.

Consider the following properties of a frame $\mathbf{F}_G$:

1) **Consistency** (**C**): $A \in \mathbf{F}_G$ implies $A^C \notin \mathbf{F}_G$.
2) **Monotonicity** (**M**): If $A \subseteq B$ and $A \in \mathbf{F}_G$, then $B \in \mathbf{F}_G$.
3) **Anonymity** (**A**): If $A \in \mathbf{F}_G$ and $\#A = \#B$, then $B \in \mathbf{F}_G$.
4) **Responsiveness** (**RE**): If $A^C \notin \mathbf{F}_G$, $A \subseteq B$ and $A \neq B$, then $B \in \mathbf{F}_G$.

A consistent frame cannot include both a coalition and its complement. When monotonic, if it includes a coalition, then it must also include any other larger coalition. An anonymous frame which includes a coalition A must also include all the coalitions with the same number of members as A. **RE** is a far cry of May's (1952) axiom. It covers two cases. First, suppose that $A \notin \mathbf{F}_G$; in conjunction with $A^C \notin \mathbf{F}_G$, we have a tie. **RE** entails that adding a new member to A we get a winning coalition B. Second, suppose that $A \in \mathbf{F}_G$. If **C** also holds, we must have that $A^C \notin \mathbf{F}_G$; then any coalition B strictly including A must also be in $\mathbf{F}_G$. (This means that **RE** in conjunction with **C** entails **M**.)

Lemma 1.

(a) If $\mathbf{F}_G$ satisfies **C** and **M**, then $\emptyset \notin \mathbf{F}_G$.

(b) If $\mathbf{F}_G$ satisfies **C** and **M**, then: if $A \in \mathbf{F}_G$ and $B \in \mathbf{F}_G$ then $A \cap B \neq \emptyset$.

(c) If $\mathbf{F}_G$ satisfies **M**, then: $A^C \notin \mathbf{F}_G$ if and only if $A \cap C \neq \emptyset$ for each $C \in \mathbf{F}_G$.

Proof. For part (a), suppose that $\mathbf{F}_G$ is non-empty. Since $\mathbf{F}_G$ satisfies **M**, we have that $G \in \mathbf{F}_G$. Given that $\mathbf{F}_G$ is consistent, we get $G^C = \emptyset \notin \mathbf{F}_G$. Moving to part (b), suppose that $A \in \mathbf{F}_G$ and $B \in \mathbf{F}_G$, but $A \cap B = \emptyset$. It follows that $A \subseteq B^C$. Since $\mathbf{F}_G$ is monotonic and $A \in \mathbf{F}_G$, we get that $B^C \in \mathbf{F}_G$. But by assumption $B \in \mathbf{F}_G$ and $\mathbf{F}_G$ is consistent, so we must have $B^C \notin \mathbf{F}_G$ – contradiction. For (c), suppose first that $A \in \mathbf{F}_G$. Then there is some C, i.e. exactly A, such that $C \in \mathbf{F}_G$ and C f A. Conversely, suppose that there is some C such that $C \in \mathbf{F}_G$ and $C \subseteq A$. Since $\mathbf{F}_G$ is monotonic and $C \in \mathbf{F}_G$, we immediately get $A \in \mathbf{F}_G$. Then A^C $\mathbf{F}_G$ if and only if it is not true that there is some $C \in \mathbf{F}_G$ such that $C \subseteq A^C$, or equivalently: for each $C \in \mathbf{F}_G$ it is false that $C \subseteq A^C$, i.e. $A \cap C \neq \emptyset$.

Observe also that **C** and **M** entail that $G \in \mathbf{F}_G$ if $\mathbf{F}_G$ is nonempty. This is equivalent with a weak Pareto condition.

The main result of this paper is this:

Theorem 1. If $\mathbf{F}_G$ satisfies properties **C**, **A** and **RE**, then $A \in \mathbf{F}_G$ iff $\#A > \#G/2$.

Proof. Suppose first that $\#G = 2k + 1$. Then we have either $\#A > k$, or $\#A^C > k$. Let $\#A > k$. We want to show that $A \in \mathbf{F}_G$. For suppose by contradiction that $A \notin \mathbf{F}_G$. We have two cases:

Case 1: $A^C \in \mathbf{F}_G$. Since we assumed that $\#A^C \leq k$ and $\mathbf{F}_G$ is monotonic we must have $B \in \mathbf{F}_G$ for all B such that $A \subseteq B$. Specifically, it must hold for some B with the property that $\#B = \#A$. But since $\mathbf{F}_G$ satisfies **A** we also get that $A \in \mathbf{F}_G$ – in contradiction with the fact that $\mathbf{F}_G$ is consistent.

Case 2: $A^C \notin \mathbf{F}_G$. There is some B such that $A^C \neq B$, $A^C \subseteq B$ and $\#B = \#A$. Given that $A \notin \mathbf{F}_G$, we must also have that $B \notin \mathbf{F}_G$. But then the responsiveness[1] of $\mathbf{F}_G$ gives: $(A^C)^C = A \in \mathbf{F}_G$.

Second, suppose that $\#G = 2k$. Now it is possible to have $\#A = \#A^C = k$. If $A \in \mathbf{F}_G$, then the fact that $\mathbf{F}_G$ satisfies **A** gives $A^C \in \mathbf{F}_G$, which contradicts the fact that $\mathbf{F}_G$ is consistent. So, neither A no A^C are not in $\mathbf{F}_G$. Let $v \in A^C$. Then $\#(A \cup \{v\}) = k + 1$. An argument analogous to the one presented above in case 2 gives that $(A \cup \{v\}) \in \mathbf{F}_G$. As noted above, since $\mathbf{F}_G$ satisfies **C** and **RE**, it also satisfies **M**. By **M** and **A**, all coalitions A with $\#A > k$ are in $\mathbf{F}_G$.

The three axioms **C**, **A** and **RE** are independent. To see this, let us construct examples of frames with the property that they make valid only two of these axioms:

Example 1: $\mathbf{F}_G = \{\emptyset\}$; **C** and **A** hold, but **RE** fails.

Example 2: $G = \{v, v'\}$ and $\mathbf{F}_G = \{\{v\}, \{v, v'\}\}$; **C** and **RE** hold, but **A** fails.

Example 3: $\mathbf{F}_G = \wp(G)$; **A** and **RE** hold, but **C** fails.

Corollary 1. If $\#A > \#A^C$, then $A \in \mathbf{F}_G$.

We have: $\#A^C = \#A + k$. So, $\#G = \#A + \#A + k = 2\#A + k$, which means that $\#A = \#G/2 + k/2 > \#G/2$. By the above theorem, $A \in \mathbf{F}_G$.

Lemma 2. If $\mathbf{F}_G$ satisfies **A** and **RE**, then: if $A \notin \mathbf{F}_G$ and also $A^C \notin \mathbf{F}_G$, then $\#A = \#A^C$.

---

[1] I appeal here to an equivalent formulation of the responsiveness axiom:

RE*: If $B \notin \mathbf{F}_G$ and A B, then $A^C \in \mathbf{F}_G$.

Proof. Let $A = D \cup \{v\}$. By **RE**, given that $A \notin \mathbf{F}_G$, $D \subseteq A$ and $D \neq A$, it follows that $D^C \in \mathbf{F}_G$. On the other hand, $A^C = (G - D) - \{v\} = D^C - \{v\}$. Therefore, $\#D^C = \#A^C + 1$. Suppose that $\#A > \#A^C$. Then there is some k such that $\#A = \#A^C + k$. So: $\#D^C = \#A - k + 1$ which entails that $\#A = \#D^C + k - 1$. So, if $k \geq 1$ we have that $\#A \geq \#D^C$. But $D^C \in \mathbf{F}_G$ and $A \notin \mathbf{F}_G$, which contradicts the fact that $\mathbf{F}_G$ is anonymous. Clearly, this corollary entails that $A \notin \mathbf{F}_G$ and also $A^C \notin \mathbf{F}_G$ can only hold if $\#G$ is odd.

With these preparations, we move to the majority rule. Observe first that in our formalism it was assumed that each voter is concerned: she either assents or dissents with an alternative. The simple majority rule is immediately characterized under this assumption. However, I shall give below a milder result, which allows for a voter to stay unconcerned.

Let $X = \{a_1, a_2\}$ be the agenda. A social choice function $c: X \rightarrow \wp(G)$ is a function which attaches a set of voters to each alternative in X. The set $c(a_i)$ is here interpreted to consist in all the voters in G who support the alternative a_i. We require that c satisfies the following two properties:

$c(a_1) \cap c(a_2) = G - H$

$v \in c(a_1)$ if and only if $v \notin c(a_2)$, for all $v \in H$.

Intuitively, we can interpret the set H as consisting in all concerned voters, i.e. voters who vote exactly one of the two alternatives. The voters in $G - H$ are unconcerned. Unconcerned voters favor both alternatives.

A majority profile on G is a quadruple $\mathbf{p}_G = (G, H, \mathbf{F}_H, c)$, where $H \subseteq G$, $\mathbf{F}_H$ is a frame on H satisfying properties **C**, **A** and **RE** and c has the two properties noted above. Say that an alternative a_i is winning at $\mathbf{p}_G$ if there is some winning coalition $A \in \mathbf{F}_G$ such that $A = c(a_i)$. The majority rule μ is defined by: the group G selects at $\mathbf{p}_G$ the alternative a_i if the number of concerned voters in it who support a_i is larger than the group of concerned voters who support the opposing alternative; otherwise G is unconcerned. Or, to put it a bit more formal, $\mu(\mathbf{p}_G) = \{a_i\}$ if $\#(H \cap c(a_i)) > \#(H \cap c(a_i)^C)$.

The following result is immediate:

Lemma 3. $\mu(\mathbf{p}_G) = \{a_i\}$ if a_i is winning at $\mathbf{p}_G$; $\mu(\mathbf{p}_G) = X$ if no alternative is winning at that profile.

Observe that if all voters are concerned, then $H = G$ and thus $\mathbf{F}_H = \mathbf{F}_G$. In this case an alternative is selected by μ if it is voted by more than half of the members of G. This also characterizes the absolute majority voting. Second, if $\mathbf{F}_G = \{G\}$, then winning coalitions define the unanimity rule.

3. On Responsiveness

Two other related results are presented below. They appeal to the notion of pick-up function. A pick-up function $\chi: \mathbf{F}_G \rightarrow G$ has the property that $\chi(A) \in A$ for each $A \in \mathbf{F}_G$. So, χ selects exactly one member of each winning coalition. Write $\chi(\mathbf{F}_G)$ for the coalition $\bigcup_{A \in \mathbf{F}_G} \chi(A)$ formed of all these voters.

Say that a frame $\mathbf{F}_G$ is χ-driven when: if $v \notin \chi(\mathbf{F}_G)$, then $\chi(\mathbf{F}_G) \cup \{v\} \in \mathbf{F}_G$; and $\mathbf{F}_G$ is driven if it is χ-driven for all χ. If we pick-up a voter from each member of a winning coalition, we either get a coalition which is winning or it becomes winning as soon as at least one other voter is added to it. The next lemma shows that this condition is equivalent to **RE** at all monotonic frames.

Lemma 4. If $\mathbf{F}_G$ is monotonic, then it is responsive if and only if it is driven.

Proof. The challenge is to show that if $\chi(\mathbf{F}_G) \notin \mathbf{F}_G$ and $v \notin \chi(\mathbf{F}_G)$, then $\chi(\mathbf{F}_G) \cup \{v\} \in \mathbf{F}_G$. First, assume that $\mathbf{F}_G$ satisfies **RE**. Let also $\chi(\mathbf{F}_G) \subseteq \chi(\mathbf{F}_G) \cup \{v\}$, but $\chi(\mathbf{F}_G) \neq \chi(\mathbf{F}_G) \cup \{v\}$. Now suppose that $(\chi(\mathbf{F}_G) \cup \{v\}) \notin \mathbf{F}_G$. Condition **RE** entails that $(\chi(\mathbf{F}_G))^C \in \mathbf{F}_G$. But then by the definition of $\chi(\mathbf{F}_G)$, there must be some $v' \in (\chi(\mathbf{F}_G))^C$ such that $v' \in \chi(\mathbf{F}_G)$ – contradiction. For the right to the left direction, assume that $\mathbf{F}_G$ is not responsive. This means that for some A and B we have that $A \subseteq B$, $A \neq B$, $A^C \notin \mathbf{F}_G$ and also $B \notin \mathbf{F}_G$. Since $A^C \notin \mathbf{F}_G$, by lemma 1c for each $C \in \mathbf{F}_G$ there is some $v \in C$ such that $v \in A$ and also $v \in B$ (because of $A \subseteq B$). Since $B \notin \mathbf{F}_G$, for each $C \in \mathbf{F}_G$ there is some $v \in C$ such that $v \notin B$ and also $v \notin A$ (because of $A \subseteq B$). Now, such an arrangement is excluded if we let $\mathbf{F}_G$ be driven. This allows us, first, to pick out of each $C \in \mathbf{F}_G$ a voter $v \in A$. In this way we get a set $A^* \subseteq A$. Secondly, we add to A^* the voters in $B - A$. Note that A^* and $B - A$ are disjoint sets. Since $\mathbf{F}_G$ is driven, we get $A^* \cup (B - A) \in \mathbf{F}_G$. But $(A^* \cup (B - A)) \subseteq B$ and since $\mathbf{F}_G$ is monotonic, we get that $B \in \mathbf{F}_G$, so $\mathbf{F}_G$ is responsive.

The reader can easily verify that if $\#G = 2k+1$, then $\chi(\mathbf{F}_G) \in \mathbf{F}_G$; but it is possible that $\chi(\mathbf{F}_G) \notin \mathbf{F}_G$ if $\#G = 2k$. For example, if $G = \{v_1, v_2, v_3, v_4\}$ and $\mathbf{F}_G = \{\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}\}$, then we can construct $\chi(\mathbf{F}_G) = \{v_1, v_2\} \notin \mathbf{F}_G$.

A final observation: for each pick-up function χ, let $G_\chi = \{v$: there are two coalitions $A \in \mathbf{F}_G$ and $B \in \mathbf{F}_G$ and $v = \chi(A \cap B)\}$. The idea is to choose a voter from each intersection of two winning coalitions. The next lemma states that the coalition of voters we get in this way is winning.

Lemma 5. If $\mathbf{F}_G$ satisfies **C** and **RE**, then $G_\chi \in \mathbf{F}_G$.

Proof. Observe first that $\chi(\mathbf{F}_G) \subseteq G_\chi$. For take some $A \in \mathbf{F}_G$. Then by definition for any $B \in \mathbf{F}_G$ there is some w such that $v \in A \cap B$ and $v \in G_\chi$, i.e. G_χ includes a member of each $B \in \mathbf{F}_G$. Now consider two coalitions B and B' with the property that there is some v such that $v \in B \subseteq B'$ and $v \notin A$. Then take $v = \chi(A \cap B)$. Clearly, $v \notin \chi(\mathbf{F}_G)$ and $v \in G_\chi$. By lemma 2, we have $\chi(\mathbf{F}_G) \cup \{v\} \in \mathbf{F}_G$. Since $(\chi(\mathbf{F}_G) \cup \{v\}) \subseteq G_\chi$ and $\mathbf{F}_G$ is monotonic, we get that $G_\chi \in \mathbf{F}_G$.

In this paper it was assumed that the set G is finite. However, the reader might have noticed the analogy between pick-up functions like χ and the so-called choice functions in set theory. On infinite domains the existence of choice functions is guaranteed in set theory by the so-called Axiom of Choice (AC). Infinite societies may represent future generations, finitely many people who extend into the indefinite future or finitely many long-lived organizations such as political parties or firms, or infinitely many voters[2]. AC was explicitly mentioned in some papers devoted to the aggregation of individual preferences on infinite domains, especially in relation to the property of Anonymity. A useful tool was the appeal to free ultrafilters. It is known that the existence of free ultrafilters can be proved by using Zorn's lemma, an equivalent of AC; see for example Brunner and Mihara (2000), Fey (2004), Cato (2019). Lemmas 4 and 5 show that on infinite domains AC is also relevant if we want to appeal to properties close to May's (1952) Responsiveness.

[2] Aumann's (1964) argument that a mathematical model appropriate to the intuitive notion of perfect competition must contain infinitely many participants is also pertinent in the field of social choice theory (Litak 2018).

References

Alcantud, J.C.R. (2020) "Simple Majorities with Voice but No Vote" Group Decision and Negotiation https://doi.org/10.1007/s10726-020-09663-5

Asan, G., Sanver, M.R. (2002) "Another characterization of the majority rule" Economics Letters **75**, 409 – 413.

Aumann, R.J. (1964) "Markets with a Continuum of Traders" Econometrica **32**, 39 – 50.

Brunner, N., Mihara, H.R. (2000) "Arrow's Theorem, Weglorz' Models and the Axiom of Choice" Mathematical Logic Quarterly **46**, 335 – 359.

Campbell, D.E., Kelly, J.S. (2000) "A simple characterization of majority rule" Economic Theory **15**, 689 – 700.

Cato, S. (2019) "The possibility of Paretian anonymous decision-making with an infinite population" Social Choice and Welfare **53**, 587 – 601.

Fey, M. (2004) "May's Theorem with an infinite population" Social Choice and Welfare **23**, 275 – 293.

Lauwers, L. (1998) "Social choice with infinite populations" in Mathematical economics by G. Chichilnisky, Ed., Northampton, Mass.: E. Elgar, 257 – 335.

Litak, T. (2018) "Infinite Populations, Choice and Determinacy" Studia Logica **10**6, 969 – 999.

May, K.O., (1952) "A set of independent, necessary and sufficient conditions for simple majority decisions" Econometrica **20**, 680 – 684.

Quesada, A. (2011) "Unconcerned groups and the majority rule" Economics Bulletin **31**, 1757- 1764.

Woeginger, G.J. (2005). "More on the majority rule: Profiles, societies, and responsiveness" Economics letters **88**, 7 – 11.

Wu, H., Xu, Y., Zhong, Z. (2015) "Single-Profile Choice Functions and Variable Societies: Characterizing Approval Voting" in Individual and Collective Choice and Social Welfare. Essays in Honor of Nick Baigent by C.Binder, G. Codognato, M. Teschl, and Y. Xu, Eds. Berlin: Springer, 143 – 147.

Xu, Y., Zhong, Z. (2010) "Single profile of preferences with variable societies: A characterization of simple majority rule" Economics Letters **107**, 119 – 121.