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Asymmetric parametric division rules revisited

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Abstract

J.E. Stovall showed that continuity, N-continuity, bilateral consistency, intrapersonal consistency, and resource monotonicity characterize division rules with continuous parametric representations. He also showed that none of the first four properties can be omitted in the characterization. In this note we continue this discussion by showing that there exists a division rule which satisfies the first four properties but not resource monotonicity. Thus these properties are independent and the formulation of Stovall´s result is optimal.

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1. INTRODUCTION

A *claims problem* is a 3-tuple (N, c, E), where $N \subset \mathbb{N}$ is a nonempty set of claimants, $c = (c_i)_{i\in N}$, where $c_i > 0$ for every $i \in N$, is a vector of claims, and $E \geq 0$ is the endowment to be divided among the claimants from the set N, which satisfies $E \leq \sum_{i\in N} c_i$. An *awards vector* for a claims problem (N, c, E) is a vector $(x_i)_{i\in N}$ such that $x_i \in [0, c_i]$ for every $i \in N$ and $\sum_{i\in N} x_i = E$. A *division rule* is a function S that maps every claims problem to an awards vector. Let us provide an example.

Example 1.1 (proportional division rule)**.** Let (N, c, E) be a claims problem. Then $S_i(N, c, E) = \lambda c_i$, $i \in N$, where $\lambda = E/\sum_{j\in N} c_j$.

Stovall (2014a) characterized a special class of division rules having a continuous parametric representation; see (Stovall, 2014a) for the precise definition. Informally speaking, for each claimant i there is a continuous monotone function, which depends on two variables; on c_i and on a parameter λ. For a given claims problem, a common parameter λ is chosen such that all of the good is distributed and the functions determine awards of each claimants. *The choice of a common parameter implies that the claimants are being treated equitably with respect to this standard of fairness* (Stovall, 2014a). This makes this class of division rules worth studying. Properties used in the characterization also reveal importance of this class.

The properties used in the formulation of the theorem are defined as follows and the definitions are followed by their informal descriptions. Interested readers are referred to (Stovall, 2014a) for motivations of the definitions below as well as for related examples.

Let S be a division rule.

Continuity. We say that S is *continuous* if for every claims problems (N, c, E), (N, c^k, E^k), $k \in \mathbb{N}$, with $(c^k, E^k) \to (c, E)$ we have $S(N, c^k, E^k) \to S(N, c, E)$.

Intrapersonal consistency. We define

$$Y = \big\{(i, c_i, x_i) \in \mathbb{N} \times (0, \infty) \times [0, \infty);\ x_i \in [0, c_i]\big\}.$$

Let $(i, c_i, x_i) \in Y$, $i, j \in \mathbb{N}$, $j \neq i$, and $c_j > 0$. We define

$$G\big((i, c_i, x_i), j, c_j\big) = \inf\big\{E;\ S_i\big(\{i, j\}, (c_i, c_j), E\big) \geq x_i\big\}.$$

The relation P_1 on Y is defined as follows. We have $(i, c_i, x_i)P_1(j, c_j, x_j)$, whenever we have

$$G\big((i, c_i, x_i), j, c_j\big) < G\big((j, c_j, x_j), i, c_i\big).$$

We say that S satisfies *intrapersonal consistency* if for every (i, c_i, x_i), (i, c'_i, x'_i), (j, c_j, x_j), $(j, c'_j, x'_j) \in Y$ such that $i \neq j$ and $(i, c_i, x_i)P_1(j, c_j, x_j)P_1(i, c'_i, x'_i)$, it is not true that

$$(i, c'_i, x'_i)P_1(j, c'_j, x'_j)P_1(i, c_i, x_i).$$

Non-comparability continuity in claims at priority points (N-continuity). We say that S *gives priority* to $(i, c_i, x_i) \in Y$ if the following two conditions are satisfied:

- $x_i \in (0, c_i)$ and
- there exists $\varepsilon > 0$ such that for every $(N, \hat{c}, E)$ where $i \in N$, $\hat{c}_i = c_i$, and $S_i(N, \hat{c}, E) = x_i$, we have $S_i(N, \hat{c}, E + \alpha) = x_i + \alpha$ whenever $\alpha \in (-\varepsilon, \varepsilon)$.

The relations R_1, R_2, and NC on Y are defined as follows. We have

- $(i, c_i, x_i)R_1(j, c_j, x_j)$, whenever
$$G\big((i, c_i, x_i), j, c_j\big) \le G\big((j, c_j, x_j), i, c_i\big).$$
- $(i, c_i, x_i)R_2(i, c'_i, x'_i)$ if there exists $(j, c_j, x_j) \in Y$ such that $j \ne i$ and
$$(i, c_i, x_i)R_1(j, c_j, x_j)R_1(i, c'_i, x'_i).$$
- $(i, c_i, x_i)NC(i, c'_i, x'_i)$ if it is neither true $(i, c_i, x_i)R_2(i, c'_i, x'_i)$ nor $(i, c'_i, x'_i)R_2(i, c_i, x_i)$.

We say that S satisfies *non-comparability continuity in claims at priority points* if for every (i, c_i, x_i) such that S gives priority to it, there exists $\varepsilon > 0$ such that for every $c'_i \in (c_i - \varepsilon, c_i + \varepsilon)$ we have $(i, c'_i, x_i)NC(i, c_i, x_i)$.

Bilateral consistency. We say that S satisfies *bilateral consistency* if for every claims problem (N, c, E) and every two-point set $N' \subset N$, we have
$$S_i(N, c, E) = S_i\Big(N', c \restriction N', \sum_{i \in N'} S_i(N, c, E)\Big), \qquad i \in N'.$$
The symbol $c \restriction N'$ denotes restriction of c to the coordinates from N'.

Resource monotonicity. We say that S satisfies *resource monotonicity* if for every claims problems (N, c, E) and (N, c, E') with $E < E'$ we have $S_i(N, c, E) \le S_i(N, c, E')$ for every $i \in N$.

Remark 1.2. Intrapersonal consistency states that the relation between different versions of claimant i will not change when the go-between's claim c_j changes. N-Continuity states that at certain situations the non-comparability of two allocations is a continuous relation with respect to small changes in the claim. Bilateral consistency states that if a division rule chooses an allocation for a set of claimants, then the awards with respect to any two-point subset does not change when considered as a separate problem. Resource monotonicity states that if the endowment E increases, then no claimant's award should decrease.

Now the characterization reads as follows.

Theorem 1.3 (Stovall (2014a))**.** *A division rule S has a continuous parametric representation if and only if S satisfies continuity, N-continuity, bilateral consistency, intrapersonal consistency, and resource monotonicity.*

Stovall also showed that none of the first four properties in the above theorem can be omitted in the characterization of continuous parametric representations and posed a problem whether one can omit resource monotonicity in the formulation of his result. The next theorem states that this is not the case. Consequently, it shows that the formulation of the characterization is optimal.

Theorem 1.4. *There exists a division rule satisfying continuity, N-continuity, bilateral consistency, intrapersonal consistency, but not resource monotonicity.*

In the next section the desired division rule is constructed and the verification of the required properties is presented in Section 3.

2. CONSTRUCTION

First we construct a division rule for problems with $N = \{1, 2\}$. Then we extend this rule to any claims problem. We define an auxiliary function $\psi\colon (0,\infty)^2 \to (0,1)$ by

$$\psi(x,y) = \frac{2x+y}{2x+2y}.$$

For $x, y \in (0,\infty)$ we define points in $\mathbb{R}^2$ by

$$\begin{aligned} &A(x,y) = [0,0], && B(x,y) = \left[\tfrac{1}{2}x, \tfrac{1}{12}y\right],\\ &C(x,y) = \left[\tfrac{1}{2}x \cdot \psi(x,y), \tfrac{1}{2}y \cdot \psi(x,y)\right], && D(x,y) = \left[\tfrac{1}{2}x, \tfrac{1}{2}y\right],\\ &E(x,y) = [x,y], && F(x,y) = [0,y],\\ &G(x,y) = [x,0]. \end{aligned}$$

For every $x, y \in (0,\infty)$ we define a mapping $\varphi^{x,y}\colon [0,1] \to \mathbb{R}^2$ by

$$\varphi^{x,y}(t) = \begin{cases} 4 \cdot B(x,y) \cdot t, & t \in [0, \frac{1}{4}],\\ 4 \cdot \big(C(x,y) - B(x,y)\big) \cdot (t - \frac{1}{4}) + B(x,y), & t \in (\frac{1}{4}, \frac{1}{2}],\\ 2 \cdot \big(E(x,y) - C(x,y)\big) \cdot (t - \frac{1}{2}) + C(x,y), & t \in (\frac{1}{2}, 1]. \end{cases}$$

The vector function $\varphi^{x,y} = [\varphi_1^{x,y}, \varphi_2^{x,y}]$ is a curve in the plane passing through points $A(x,y)$, $B(x,y)$, $C(x,y)$, $D(x,y)$, and $E(x,y)$. See Figure 1. Observe that the function $\Phi^{x,y}\colon [0,1] \to \mathbb{R}$

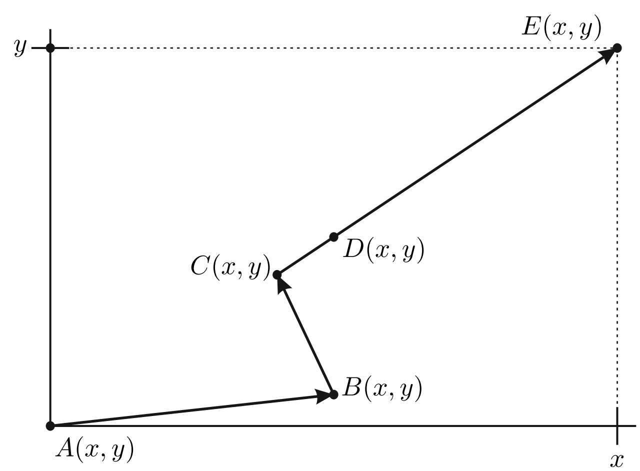

FIGURE 1

defined for any $x, y \in (0,\infty)$ by $\Phi^{x,y}(t) = \varphi_1^{x,y}(t) + \varphi_2^{x,y}(t)$, $t \in [0,1]$, is strictly increasing on $[0,1]$. Indeed, $\Phi^{x,y}$ is continuous on $[0,1]$ and a straightforward computation gives that the derivative

$$(\Phi^{x,y})'(t) = \begin{cases} 2x + \frac{1}{3}y, & t \in (0, \frac{1}{4}),\\ \frac{2}{3}y, & t \in (\frac{1}{4}, \frac{1}{2}),\\ x + \frac{3}{2}y, & t \in (\frac{1}{2}, 1], \end{cases}$$

is positive for every $t \in (0,1) \setminus \{\frac{1}{4}, \frac{1}{2}\}$.

Now we define a solution $\tilde{S}$ for the claims problems $\big(\{1,2\}, c_1, c_2, E\big)$ as follows

$$\tilde{S}_1(c_1, c_2, E) = \varphi_1^{c_1,c_2}(t^*), \qquad \tilde{S}_2(c_1, c_2, E) = \varphi_2^{c_1,c_2}(t^*),$$

where $t^* \in (0,1]$ is the unique real number satisfying $\Phi^{c_1,c_2}(t^*) = E$. Such a t^* exists since Φ^{c_1,c_2} is strictly increasing, continuous on $[0,1]$, and $\Phi^{c_1,c_2}(0) = 0 \le E \le c_1 + c_2 = \Phi^{c_1,c_2}(1)$.

Now we extend $\tilde{S}$ to S which will be defined on the domain of all claims problems. Let (N, c, E) be a claims problem. If $1, 2 \notin N$, then we use the proportional rule from Example 1.1. If this is not the case, then the rule S satisfies first the claims of claimants 1 and 2, whereas $\tilde{S}$ is used if $1, 2 \in N$, and then the reminder of the endowment is split among the other claimants using the proportional rule again.

The core of the counterexample is captured in the definition of $\tilde{S}$. Willing to break resource monotonicity we define a path connecting $[0,0]$ and $[c_1, c_2]$ in such way that the path contains points B and C placed as in Figure 1 and the rest of the path is chosen as simple as possible, i.e., it is piecewise-affine. The position of the points B and C is arranged using the function ψ so that if the point $[u, v]$ follows the path then the function $u + v$ is increasing despite of the kink among points B, C, and D. This property enables us to split the endowment uniquely in the claims problem $\big(\{1,2\}, (c_1, c_2), E\big)$.

3. PROOF

We stepwise verify that the above define division rule S satisfies continuity, N-continuity, bilateral consistency, intrapersonal consistency and does not satisfy resource monotonicity.

Continuity of S. First we check continuity of $\tilde{S}$. Then the continuity of S easily follows. Suppose that $(c_1^k, c_2^k, E^k) \to (c_1, c_2, E)$, where $(\{1,2\}, (c_1^k, c_2^k), E^k)$, $k \in \mathbb{N}$, and $(\{1,2\}, (c_1, c_2), E)$ are claims problems. Find t^k, $k \in \mathbb{N}$, and t^* from $[0,1]$ such that

$$\tilde{S}_i(c_1^k, c_2^k, E^k) = \varphi_i^{c_1^k, c_2^k}(t^k), \qquad \tilde{S}_i(c_1, c_2, E) = \varphi_i^{c_1,c_2}(t^*), \qquad i \in \{1,2\}.$$

We may assume without any loss of generality that

(a) either for every $k \in \mathbb{N}$ we have $t^k \in (0, \frac{1}{4}]$ or
(b) for every $k \in \mathbb{N}$ we have $t^k \in (\frac{1}{4}, \frac{1}{2})$ or
(c) for every $k \in \mathbb{N}$ we have $t^k \in (\frac{1}{2}, 1]$.

We discuss these cases separately.

(a) In this case we have

$$t^k = \frac{E^k}{2c_1^k + \frac{1}{3}c_2^k} \to \frac{E}{2c_1 + \frac{1}{3}c_2} = t^* \in (0, \tfrac{1}{4}].$$

Consequently,

$$\tilde{S}_i(c_1^k, c_2^k, E^k) = \varphi_i^{c_1^k, c_2^k}(t^k) \to \varphi_i^{c_1,c_2}(t^*) = \tilde{S}_i(c_1, c_2, E), \qquad i \in \{1,2\}.$$

(b) Using continuity of ψ we have

$$t^k = \frac{E^k - \frac{1}{2}c_1^k - \frac{1}{12}c_2^k}{2(c_1^k + c_2^k)\psi(c_1^k, c_2^k) - 2c_1^k - \frac{1}{3}c_2^k} + \frac{1}{4} \to \frac{E - \frac{1}{2}c_1 - \frac{1}{12}c_2}{2(c_1 + c_2)\psi(c_1, c_2) - 2c_1 - \frac{1}{3}c_2} + \frac{1}{4} = t^* \in [\tfrac{1}{4}, \tfrac{1}{2}].$$

Consequently,

$$\tilde{S}_i(c_1^k, c_2^k, E^k) = \varphi_i^{c_1^k, c_2^k}(t^k) \to \varphi_i^{c_1, c_2}(t^*) = \tilde{S}_i(c_1, c_2, E), \qquad i \in \{1, 2\}.$$

(c) This case can be handled in the same way as in (b).

N-continuity of S.

Lemma 3.1. *The rule S gives priority to no $(i, c_i, x_i) \in Y$.*

Proof. Let $(i, c_i, x_i) \in Y$ be such that $x_i \in (0, c_i)$. If $i = 1$, then we find $\hat{c}_2 > 0$ and $E > 0$ such that $S_1(\{1, 2\}, \hat{c}, E) = x_i = x_1$, where $\hat{c} = (c_1, \hat{c}_2)$. For every sufficiently small $\alpha > 0$ we have $S_1(\{1, 2\}, \hat{c}, E + \alpha) < x_1 + \alpha$. Thus there is no $(1, c_1, x_1)$ such that S gives priority to it. The reasoning for $i = 2$ is similar.

If $i > 2$ then we consider a claim problem $(N, \hat{c}, E)$ defined by $N = \{i, i + 1\}$, $E = 2x_i$, $\hat{c}_i = \hat{c}_{i+1} = c_i$. Then $S_i(N, \hat{c}, E) = x_i$. If $\alpha > 0$ is sufficiently small then $S_i(N, \hat{c}, E + \alpha) = \frac{1}{2}(E + \alpha) = x_i + \frac{1}{2}\alpha < x_i + \alpha$. Thus also in this case S does not give priority to (i, c_i, x_i). □

From Lemma 3.1 it follows that S satisfies N-continuity trivially.

Bilateral consistency of S. Let (N, c, E) be a claims problem and $N' = \{i, j\} \subset N$ be a two-point set. We distinguish several possibilities.

The case $N' = \{1, 2\}$. We have

$$\begin{aligned} S_k(N, c, E) &= \tilde{S}_k(c_1, c_2, \min\{c_1 + c_2, E\}), \qquad k \in \{1, 2\}, \\ S_k(N', (c_1, c_2), S_1(N, c, E) + S_2(N, c, E)) &= S_k(N', (c_1, c_2), \min\{c_1 + c_2, E\}) \\ &= \tilde{S}_k(c_1, c_2, \min\{c_1 + c_2, E\}), \qquad k \in \{1, 2\}. \end{aligned}$$

Thus we have the desired equality

$$S_k(N, c, E) = S_k(N', (c_1, c_2), S_1(N, c, E) + S_2(N, c, E)), \qquad k \in \{1, 2\} = N'.$$

The case $i = 1 \in N'$ and $2 \notin N'$. We have

$$S_1(N, c, E) = \min\{c_1, E\},$$

$$S_j(N, c, E) = \lambda c_j, \text{ where } \lambda \cdot \sum_{l \in N, l \neq 1} c_l = E - \min\{c_1, E\},$$

$$S_1(N', (c_1, c_j), \min\{c_1, E\} + \lambda c_j) = \min\{c_1, \min\{c_1, E\} + \lambda c_j\} = \min\{c_1, E\},$$

$$S_j(N', (c_1, c_j), \min\{c_1, E\} + \lambda c_j) = \lambda c_j.$$

Thus we get

$$S_k(N, c, E) = S_k\big(N', (c_1, c_j), S_1(N, c, E) + S_j(N, c, E)\big), \qquad k \in \{1, j\}.$$

The other cases can be handled in the same way and we will not present them explicitly.

Intrapersonal consistency of S. We start with the following notation.

Notation 3.2. Let $X, Y \in \mathbb{R}^2$. Then the symbol $[X, Y]$ denotes the line segment in the plane with endpoints X and Y, the symbol $(X, Y]$ stands for the line segment $[X, Y] \setminus \{X\}$. The meaning of the symbols $[X, Y)$ and (X, Y) is now obvious.

Lemma 3.3. *Let $c_1, c_2 > 0$. Then the set*

$$Q_1(c_1, c_2) = \big\{(x_1, x_2) \in [0, c_1] \times [0, c_2];\ G\big((1, c_1, x_1), 2, c_2\big) < G\big((2, c_2, x_2), 1, c_1\big)\big\}$$

is the polygon with the vertices $A(c_1, c_2)$, $B(c_1, c_2)$, $D(c_1, c_2)$, $E(c_1, c_2)$, and $F(c_1, c_2)$ such that the line segments $(B(c_1, c_2), D(c_1, c_2)]$, $(E(c_1, c_2), F(c_1, c_2)]$, and $[F(c_1, c_2), A(c_1, c_2))$ are subsets of $Q_1(c_1, c_2)$ and other points of the boundary of $Q_1(c_1, c_2)$ do not belong to $Q_1(c_1, c_2)$. See Figure 2.

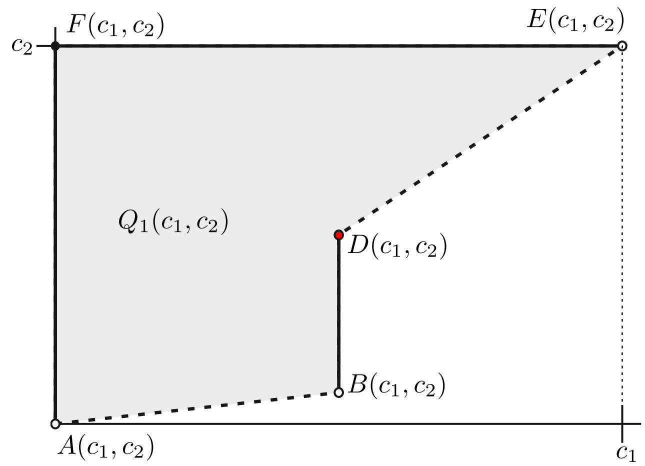

FIGURE 2

Lemma 3.4. *Let $c_1, c_2 > 0$. Then the set*

$$Q_2(c_1, c_2) = \big\{(x_1, x_2) \in [0, c_1] \times [0, c_2];\ G\big((1, c_1, x_1), 2, c_2\big) > G\big((2, c_2, x_2), 1, c_1\big)\big\}$$

is the polygon with the vertices $A(c_1, c_2)$, $B(c_1, c_2)$, $D(c_1, c_2)$, $E(c_1, c_2)$, and $G(c_1, c_2)$ such that the line segments $(E(c_1, c_2), G(c_1, c_2)]$, and $[G(c_1, c_2), E(c_1, c_2))]$ are subsets of $Q_2(c_1, c_2)$ and other points of the boundary of $Q_2(c_1, c_2)$ do not belong to $Q_2(c_1, c_2)$. See Figure 3.

Both Lemmas can be inferred by discussing position of points (x_1, x_2) in the rectangle $[0, c_1] \times [0, c_2]$. We omit these straightforward proofs.

Lemma 3.5. *Let $c_1, c_2 > 0$. If $(x_1, x_2) \in Q_1(c_1, c_2)$ and $(x_1, x_2') \in Q_2(c_1, c_2)$, then $x_2' < x_2$.*

Proof. Using Lemma 3.3 we have that for every $x_1 \in [0, c_1]$ the set $\big\{z \in [0, c_2];\ (x_1, z) \in Q_1(c_1, c_2)\big\}$ is an interval of the form $(\alpha, c_2]$. This implies the assertion. ☐

Lemma 3.6. *Let $c_1, c_2 > 0$ and $\alpha > 0$. If $(x_1, x_2) \in Q_1(c_1, c_2)$, then $(x_1, \alpha x_2) \in Q_1(c_1, \alpha c_2)$. Similarly, if $(x_1, x_2) \in Q_2(c_1, c_2)$, then $(x_1, \alpha x_2) \in Q_2(c_1, \alpha c_2)$.*

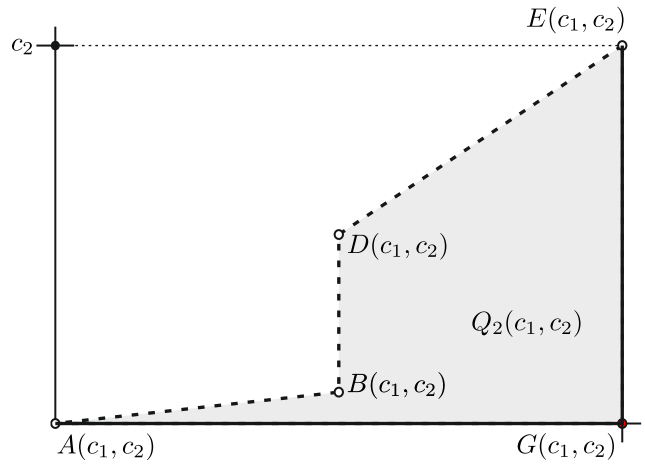

FIGURE 3

Proof. Fix $\alpha > 0$. Let $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping defined by $L(x, y) = (x, \alpha y)$. We want to prove that $L(Q_1(c_1, c_2)) = Q_1(c_1, \alpha c_2)$. The mapping L is linear and maps points $A(c_1, c_2)$, $B(c_1, c_2)$, $D(c_1, c_2)$, $E(c_1, c_2)$, and $F(c_1, c_2)$ to the points $A(c_1, \alpha c_2)$, $B(c_1, \alpha c_2)$, $D(c_1, \alpha c_2)$, $E(c_1, \alpha c_2)$, and $F(c_1, \alpha c_2)$ respectively. By Lemma 3.3 this easily implies the desired equality. The reasoning in the case of $Q_2(c_1, c_2)$ is analogous. $\square$

Now we verify intrapersonal consistency of S. Towards contradiction assume that

$$(i, c_i, x_i) P_1 (j, c_j, x_j) P_1 (i, c_i', x_i') \quad \text{and} \quad (i, c_i', x_i') P_1 (j, c_j', x_j') P_1 (i, c_i, x_i). \tag{1}$$

We distinguish several possibilities.

The case $i = 1, j = 2$. From (1) we have

$$(x_1, x_2) \in Q_1(c_1, c_2), \tag{2}$$
$$(x_1', x_2) \in Q_2(c_1', c_2), \tag{3}$$
$$(x_1', x_2') \in Q_1(c_1', c_2'), \tag{4}$$
$$(x_1, x_2') \in Q_2(c_1, c_2'). \tag{5}$$

Using Lemma 3.6, (4), and (5) we get

$$\left(x_1', \tfrac{c_2}{c_2'} x_2'\right) \in Q_1(c_1', c_2), \tag{6}$$
$$\left(x_1, \tfrac{c_2}{c_2'} x_2'\right) \in Q_2(c_1, c_2). \tag{7}$$

Using Lemma 3.5, (3), and (6) we get $x_2 < \frac{c_2}{c_2'} x_2'$. Using Lemma 3.5, (2), and (7) we get $x_2 > \frac{c_2}{c_2'} x_2'$, a contradiction.

The case $i = 2, j = 1$. We have

$$(x_1, x_2) \in Q_2(c_1, c_2), \tag{8}$$

$$(x_1, x'_2) \in Q_1(c_1, c'_2), \tag{9}$$

$$(x'_1, x'_2) \in Q_2(c'_1, c'_2), \tag{10}$$

$$(x'_1, x_2) \in Q_1(c'_1, c_2). \tag{11}$$

Using Lemma 3.6, (9), and (10) we get

$$(x_1, \tfrac{c_2}{c'_2} x'_2) \in Q_1(c_1, c_2), \tag{12}$$

$$(x'_1, \tfrac{c_2}{c'_2} x'_2) \in Q_2(c'_1, c_2). \tag{13}$$

Using Lemma 3.5, (8), and (12) we get $x_2 < \frac{c_2}{c'_2} x'_2$. Using Lemma 3.5, (11), and (13) we get $x_2 > \frac{c_2}{c'_2} x'_2$, a contradiction.

The case $i \in \{1, 2\}$ *and* $j \notin \{1, 2\}$. Then we have $x_j = 0$ or $(i, c'_i, x'_i) P_1 (j, c_j, x_j)$. Both possibilities lead to a contradiction with the assumption.

The case $j \in \{1, 2\}$ *and* $i \notin \{1, 2\}$. Then we have $x_i = 0$ or $(j, c_j, x_j) P_1 (i, c_i, x_i)$. Both possibilities lead to a contradiction with the assumption.

The case $i \notin \{1, 2\}$ *and* $j \notin \{1, 2\}$. The first part of (1) gives $\frac{x_i}{c_i} < \frac{x_j}{c_j} < \frac{x'_i}{c'_i}$. The second part gives $\frac{x'_i}{c'_i} < \frac{x'_j}{c'_j} < \frac{x_i}{c_i}$. Together we have $\frac{x_i}{c_i} < \frac{x_i}{c_i}$, a contradiction.

Falsity of resource monotonicity for S**.** We have $B(1, 1) = [\frac{1}{2}, \frac{1}{12}]$ and $C(1, 1) = [\frac{3}{8}, \frac{3}{8}]$. Therefore $S_1(\{1, 2\}, (1, 1), \frac{7}{12}) = B(1, 1)_1 = \frac{1}{2}$ and $S_1(\{1, 2\}, (1, 1), \frac{6}{8}) = C(1, 1)_1 = \frac{3}{8}$. This shows that S does not satisfy resource monotonicity.

Remark 3.7. Stovall (2014b, Appendix B) constructed a special division rule which satisfies certain axioms but does not satisfy resource monotonicity. This division rule provides another counterexample. Stovall's division rule is defined using a family of functions, which all but one satisfy the properties required in the definition of asymmetric parametric division rule. The exceptional function is not even monotone, and therefore the division rule does not satisfy resource monotonicity. On the other hand the other axioms from Stovall's characterization can be verified.

In this paper, we presented an approach which is rather geometrical than analytical and provides another intuitive insight into the behaviour of the relation P_1.

REFERENCES

Stovall, J. E. (2014a). "Asymmetric parametric division rules". *Games Econom. Behav.*, 84:87–110.

Stovall, J. E. (2014b). "Collective rationality and monotone path division rules". *J. Econ. Theory*, 154:1–24.