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Ambiguity-aversion in a Single Auction Market

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Abstract

Within Kyle's single auction model, we show that an ambiguity-averse insider, who is uncertain about the market maker's beliefs, implements a robust trading strategy, so that she selects as her market order that which maximizes her expected profits against those beliefs which penalize her most. Her trading strategy is equivalent to that of a risk-averse insider who does not face any Knightian uncertain. As she finds it optimal to trade less aggressively and reveal her private information at a slower pace than her risk-neutral counterpart, ambiguity-aversion reduces market efficiency but improves market liquidity.

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1 Kyle's Single Auction Model with Ambiguity-aversion

In Kyle's (Kyle, 1985) single auction model the insider maximizes the expected value of her trading profits. These are $\pi = (v - p)x$, where x is her market order, v the liquidation value of the risky asset, with $v \sim N(p_0, \Sigma_0)$, and p the corresponding transaction price set by the market maker.[1]

The insider observes v before trading occurs. She acts strategically and chooses her market order to maximize the expected value of her profits. The market maker sets the transaction price according to a semi-strong form efficiency condition, so that $p = p_0 + \lambda(x + u)$ with λ a positive coefficient and u the market order of the liquidity traders, with $u \sim N(0, \sigma_u^2)$.

We extend Kyle's analysis by investigating a scenario in which the insider is uncertain about the market maker's beliefs and the pricing rule he applies. We deem this scenario worth analyzing because in many real situations some investors may have privileged information on securities' fundamentals, but limited knowledge of the activity and beliefs of other market participants. Specifically, in equity markets main shareholders or senior managers may have privileged access to companies' information and a better understanding of their fundamentals. However, these insiders are likely to possess only limited knowledge of the activity of other traders and of the mechanisms which dictate how these market participants form their beliefs about stocks' fundamentals. Therefore, it is likely that they are uncertain about the process dealers follow in setting transaction prices in equity markets.

1.1 Knightian Uncertainty

We then assume that the insider is uncertain about the pricing rule used by the market maker, as she is unsure about the mechanisms which regulate how he forms his beliefs about the liquidation value. Thus, she suspects that the market maker does not set the transaction price according to the semi-strong efficiency condition above, in that $p = p_0 + \lambda(x + u) + \epsilon$ with ϵ an unspecified value. For $\epsilon \neq 0$ the transaction price is biased, possibly because the market maker ignores some predictable component of liquidity trading or because he manages his inventory of the risky asset. However,

[1]As customary in the literature the insider is refereed to as "she" and the market maker as "he".

because ϵ is unspecified, the insider cannot calculate the exact probability distribution of the profits any trading decision will generate, so that her uncertainty corresponds to Knightian uncertainty.

In our treatment of Knightian uncertainty we follow Hansen and Sargent's approach (Hansen and Sargent, 2008).Therefore, we assume that the insider conjectures that the mis-pricing of the risky asset respects the following *approximating* specification

$$v - p = v - p_0 - \lambda(x + u), \tag{1}$$

while she suspects that it actually respects the *distorted* specification

$$v - p = v - p_0 - \lambda(x + u) + \sigma_u w, \text{ with } \sigma_u w = -\frac{1}{\lambda}\epsilon \tag{2}$$

an undetermined value. The probabilistic distance between these two specifications is measured by the expected value, under the distorted specification, of the log of the ratio between the conditional probability density functions for $z \equiv v - p$ under the distorted and the approximating specifications. Analytically, the *relative entropy* is

$$I(f_a, f_d) \equiv \int \log(f_d(z)/f_a(z))f_d(z)dz,$$

with $f_m(z)$ the probability density function of z conditionally on observing v under specification m. Because shocks are normally distributed, under the approximating specification $z \mid v \sim N(v - p_0 - \lambda x, \lambda^2\sigma_u^2)$, while under the distorted one $z \mid v \sim N(v - p_0 - \lambda x + \lambda\sigma_u w, \lambda^2\sigma_u^2)$. The following Lemma holds.

Lemma 1 *The relative entropy is* $I(f_a, f_d) = \frac{1}{2}w^2$.

Proof. For $\sigma_z = \lambda\sigma_u$ and $\mu_z = v - p_0 - \lambda x$,

$$f_a(z) = \frac{1}{\sqrt{\pi}\sigma_z}\exp\left[-\frac{1}{2}\left(\frac{z-\mu_z}{\sigma_z}\right)^2\right] \quad \text{and} \quad f_d(z) = \frac{1}{\sqrt{\pi}\sigma_z}\exp\left[-\frac{1}{2}\left(\frac{z-\mu_z-\sigma_z w}{\sigma_z}\right)^2\right].$$

Thus,

$$\frac{f_d(z)}{f_a(z)} = \exp\left[\frac{1}{2}\frac{1}{\sigma_z^2}\Big((z-\mu_z)^2 - \Big((z-\mu_z-\sigma_z w)^2\Big)\Big]\right].$$

Since $(z-\mu_z)^2 \;=\; (z-\mu_z-\sigma_z w)^2 \,+\, \sigma_z^2 w^2 \,+\, 2\sigma_z w(z-\mu_z-\sigma_z w)$,

$$\frac{f_d(z)}{f_a(z)} \;=\; \exp\left[\frac{1}{2}w^2 \,+\, \frac{w}{\sigma_z}(z-\mu_z-\sigma_z w)\right] .$$

Then, $\log\left(\frac{f_d(z)}{f_a(z)}\right) \;=\; \frac{1}{2}w^2 \,+\, \frac{w}{\sigma_z}(z-\mu_z-\sigma_z w)$, so that

$$I(f_a,f_d) \;=\; \int \log\left(\frac{f_d(z)}{f_a(z)}\right) f_d(z)dz \;=\; \int \frac{1}{2}w^2\, f_d(z)dz + \frac{w}{\sigma_z}\int (z-\mu_z-\sigma_z w)\, f_d(z)dz \;=\; \frac{1}{2}w^2 \,.\square$$

1.2 Ambiguity-aversion and Robust Trading Strategies

We assume that an ambiguity-averse insider facing the Knightian uncertainty defined above solves the following *constraint problem*

$$\begin{aligned} &\max_x \min_w E[\pi]\,, \\ &\text{s.t. } v-p \;=\; v \;-\; p_0 \;-\; \lambda\,(x\,+\,u) \;+\; \sigma_u\, w \\ &\text{and } I(f_a,f_d) \;\le\; \phi\,, \end{aligned} \tag{3}$$

with ϕ a positive value representing the maximum probabilistic distance between the approximating and distorted specifications she deems feasible.

According to this problem, the insider considers all possible alternative distorted specifications which are, probabilistically, not too far from the approximating one. She then selects that market order which maximizes her expected trading profits against the *worst* choice of w, ie. for the distorted specification which is most detrimental to her profit opportunities. The ambiguity-averse insider applies this particularly restrictive selection criterion, which identifies a *robust* trading strategy, because it allows her to deal with her inability to calculate the probability of the outcomes of her trading activity.

Maccheroni *et al.* (2006) show (see Proposition 19) that the preferences of an agent solving the constraint problem above are equivalent those of an agent solving the following *multiplier problem*

$$\max_x \min_w E\left[\pi + \frac{1}{\theta} w^2\right], \tag{4}$$

$$\text{s.t. } v - p = v - p_0 - \lambda(x + u) + \sigma_u w,$$

for a particular positive penalty parameter θ restraining the minimization choice of w. This parameter defines the insider's degree of ambiguity-aversion.

1.3 Market Equilibrium

To identify the market equilibrium, consider that the insider observes the liquidation value v before selecting her market order x. She knows that the market maker is risk-neutral and is forced to set the transaction price for the risky asset according to a semi-strong efficiency condition, because of potential Bertrand competition by other dealers. The insiders also knows that the market maker in setting this price uses his initial prior on the liquidation value v plus the information he can extract form order flow, $x + u$. On his part, the market maker knows that unconditionally $v \sim N(p_0, \Sigma_0)$ and that $u \sim N(0, \sigma_u^2)$. In addition, before setting p the market maker observes the overall market order $x + u$. Finally we assume the market maker knows the insider's preferences and her degree of ambiguity-aversion. This implies he knows that the insider in selecting her market order solves the constraint problem (4).

In brief, the insider's trading strategy, $x = X(v)$, solves the constraint problem (4), while the market maker's pricing rule, $p = P(x + u)$, respects the efficiency condition $p = E[v \mid x + u]$. A market equilibrium is a couple $\langle X, P \rangle$ which satisfies both conditions. The following Proposition holds.

Proposition 1 *A unique linear equilibrium for the market for the risky asset exists. In this equilibrium the insider's market order and the price set by the market maker are*

$$x = \beta(v - p_0), \tag{5}$$

$$p = p_0 + \lambda(x + u), \tag{6}$$

with the coefficients β and λ the unique positive solutions to the system

$$\beta = \frac{1}{\lambda\left(2 + \frac{1}{2}\lambda\theta\sigma_u^2\right)} \quad \textit{and} \quad \lambda = \frac{\beta\Sigma_0}{\beta^2\Sigma_0 + \sigma_u^2}. \tag{7}$$

Proof. Assume $p = p_0 + \lambda(x+u)$. To solve problem (4) we rely on the certainty equivalence setting $u = 0$ and consider the following *extremization*

$$\max_x \min_w \left[(v - p_0 - \lambda x + \lambda\sigma_u\, w)\, x + \frac{1}{\theta} w^2 \right].$$

The argmin with respect to w is $\breve{w} = -\frac{\theta}{2}\,\lambda\sigma_u\, x$. The corresponding second order condition is $\frac{2}{\theta} > 0$, which is always satisfied. In the second stage extremization we solve

$$\max_x \left[(v - p_0 - \lambda x)\, x - \frac{\theta}{4}(\lambda\sigma_u)^2 x^2 \right]. \tag{8}$$

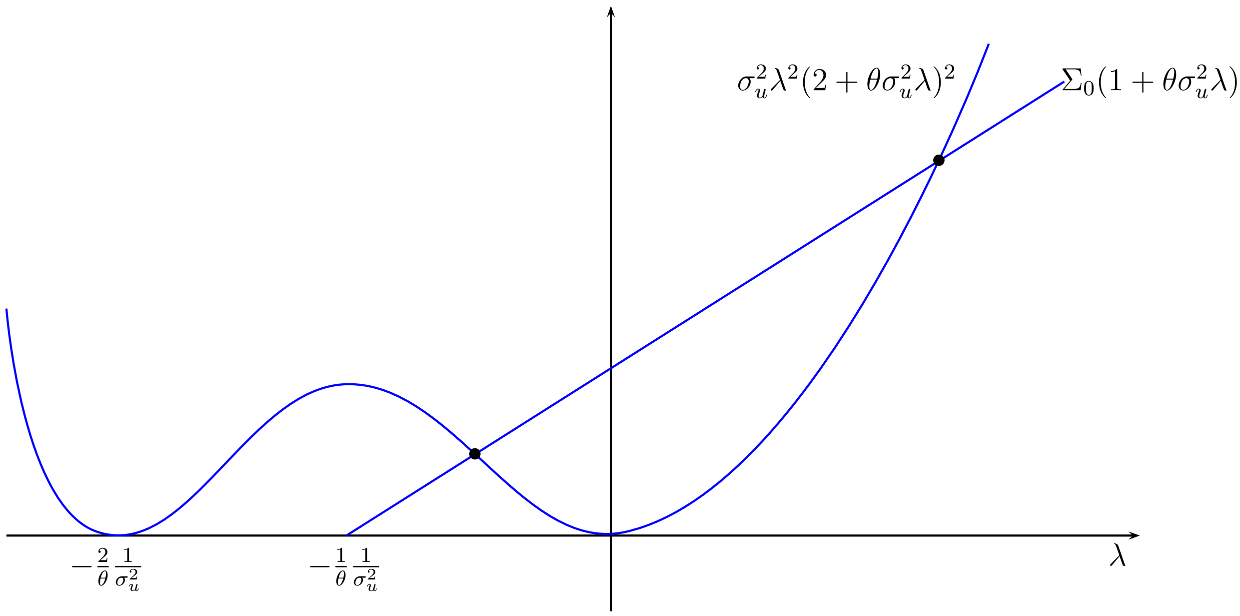
The argmax is $x = \beta(v - p_0)$ with $\beta = \frac{1}{\lambda\left(2+\frac{1}{2}\lambda\theta\sigma_u^2\right)}$ and second order condition $\lambda(2 + \frac{1}{2}\lambda\theta\sigma_u^2) > 0$.

Suppose that $x = \beta(v - p_0)$. Applying the projection Theorem for Normal distributions we find that $E[v \mid x+u] = p_0 + \lambda(x+u)$, with $\lambda = \frac{\beta\Sigma_0}{\beta^2\Sigma_0 + \sigma_u^2}$.

To prove unicity consider that inserting the expression for β in λ we find that λ must satisfy

$$\sigma_u^2\, \lambda^2\, (2 + \theta\, \sigma_u^2\, \lambda)^2 \;=\; (1 + \theta\, \sigma_u^2\, \lambda)\, \Sigma_0\, .$$

This equation possesses a negative and a positive solution, as shown by the diagram below.



However, the negative one implies that $\lambda(1+\theta\sigma_u^2\lambda)<0$ and $\lambda(2+\theta\sigma_u^2\lambda)<0$, violating the second order condition for the insider's second stage extremization, $\lambda(2+\theta\sigma_u^2\lambda)>0$. $\square$

2 Ambiguity-aversion vs Risk-aversion

We now establish an equivalence between the impact of ambiguity-aversion and that of risk-aversion on the trading strategy of the insider and on the characteristics of the market for the risky asset. In fact, assume as in Subrahmanyam (1991) that the insider does not face any Knightian uncertainty on the market maker's pricing rule and that she is endowed with a CARA utility function of her trading profits with coefficient of absolute risk-aversion ρ. The following Proposition holds.

Proposition 2 *When the insider faces no Knightian uncertainty and possesses a CARA utility function, the market equilibrium coincides with that described in Proposition 1 for ρ replacing θ.*

Proof. Given her CARA utility function with coefficient of absolute risk-aversion ρ, the insider maximizes $E[\pi \mid v]-\frac{1}{2}\rho\text{Var}[\pi \mid v]$. Under the linear pricing rule $p=p_0+\lambda(x+u)$, she chooses x by maximizing $(v-p_0-\lambda x)x-\frac{1}{2}\rho\lambda^2\sigma_u^2x^2$ which corresponds to the second stage extremization in (8) the insider solves under ambiguity-aversion. $\square$

Proposition 2 implies that ambiguity- and risk-aversion have a similar impact on the insider's trading strategy and on market quality even if they represent different characterizations of agents' attitude towards risk. In fact, under the former the insider is concerned with the volatility of her profits, while under the latter she is also uncertain about their expected value.

2.1 The Impact of Ambiguity-aversion

Corollary of Proposition 2 is the following result.

Corollary 1 *Market liquidity (efficiency) is increasing (decreasing) in the degree of ambiguity-aversion θ.*

Proof. Plugging the expression for λ into that for β in the system (7) we find that β solves the following equation

$$\Sigma_0^2\,\beta^4\,+\,\Sigma_0\,\theta\sigma_u^2\,\beta^3\;=\;\sigma_u^4\,.$$

Let $F(\beta,\theta)\equiv\Sigma_0^2\,\beta^4\,+\,\Sigma_0\,\theta\sigma_u^2\,\beta^3-\sigma_u^4$ and consider that for $\beta>0$, $\frac{\partial F}{\partial\theta}>0$ and $\frac{\partial F}{\partial\beta}>0$ and hence $\frac{\partial\beta}{\partial\theta}=-\frac{\partial F}{\partial\theta}\Big/\frac{\partial F}{\partial\beta}<0$. Then, since $\lambda(\beta)=\frac{\beta\Sigma_0}{\beta^2\Sigma_0+\sigma_u^2}$, we see that $\frac{\partial\lambda}{\partial\beta}>0$ insofar $\beta<\bar{\beta}\equiv\left(\frac{\sigma_u^2}{\Sigma_0}\right)^{1/2}$. Now,

$$\left(\frac{1}{\beta}\right)^2\;=\;\lambda^2(2+\theta\sigma_u^2\lambda)^2=(1+\theta\sigma_u^2\lambda)\frac{\Sigma_0}{\sigma_u^2}\,.$$

Since θ and λ are positive, $\left(\frac{1}{\beta}\right)^2>\frac{\Sigma_0}{\sigma_u^2}\Leftrightarrow\beta<\bar{\beta}$. Then, applying the chain rule we find that $\frac{\partial\lambda}{\partial\theta}=\frac{\partial\lambda}{\partial\beta}\times\frac{\partial\beta}{\partial\theta}<0$. Market liquidity is measured by $1/\lambda$, which is then increasing in θ. Market efficiency is given by the inverse of the market maker's conditional variance of v. From the projection Theorem for Normal distributions we find that this variance is $\Sigma_1=(1-\lambda\beta)\Sigma_0$. Because β and λ are decreasing in θ, efficiency is decreasing in θ. $\square$

Because for $\theta=0$ the market equilibrium in Proposition 1 collapses to that unveiled by Kyle for the scenario with a risk-neutral insider, this Corollary implies that market liquidity (efficiency) is larger (smaller) with an ambiguity-averse insider than with a risk-neutral one.

We conclude that in Kyle's single auction market, ambiguity-aversion makes the insider trade less aggressively. In this way she reveals a smaller fraction of her informational advantage, while the market turns out to be less efficient but more liquid. Our results are consistent with the portfolio inertia exhibited in models of asset prices with ambiguity-averse investors, as shown among others by Dow and Da Costa Werlang (1992).

Interestingly, while very little empirical research on the impact of ambiguity-aversion on the performance of securities markets, evidence by Dimmock *et al.* (2016) suggests that ambiguity-averse agents limit their participation to equity markets. As limited participation reduces market efficiency this evidence confirms the claims made in Corollary 1.

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