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Recursive methods for discrete claims problems

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Abstract

We consider claims problems with indivisible goods. Specifically, by applying the recursive procedures proposed by Giménez-Gomez and Marco-Gil (2014) and Giménez-Gómez and Peris (2014), we ensure the fulfilment of order preservation and balancedness, considered by many authors as minimal requirements of fairness. Moreover, we retrieve the discrete constrained equal losses and the discrete constrained equal awards rules (Herrero and Martinez, 2008). By the recursive imposition of a lower bound and an upper bound, we obtain the average between them.

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1. Introduction

When a firm goes bankrupt, and the assets are not enough to satisfy its creditors' demands, how should those assets be divided up among the creditors? This important issue of claims problems acquires a special interest in the current global financial crisis. In this kind of situations the amount to allocate and the demands of the creditors are perfectly divisible and homogeneous. However, there are many real-world situations in which both the amount to allocate and the claims are indivisible and identical units. Thus, every solution will assign an integer of units of this good to each agent. Consider, for instance, the cases of waiting lists for surgery at hospitals, airport slots demanded by airlines, and allocations of visas to potential immigrants. The data of these problems, as well as the allocations, are all integers. Even when the demands can be satisfied, discreteness may be a problem (see Balinski and Young, 2001).

Situations like these, are studied under the framework of the so-called claims problem, originated by O'Neill (1982), either for perfectly divisible claims (see Bergantiños et al., 2010; Giménez-Gómez and Peris, 2014b; Bergantiños and Moreno-Ternero, 2016; among others), or for the indivisible case.[1] Usually, in the case of indivisible claims problems, priority (rationing) methods are applied (Moulin, 2000; Herrero and Martínez, 2008b; and Chen, 2015). Specifically, in rationing problems when agents have single peaked preferences, Herrero and Martínez (2008b) consider the M-down method which can be interpreted as the discrete version of the well-known constrained equal awards rule in the divisible case. Nonetheless, applying the M-down method to the claims problems yields a less desirable rule, and thus the claims problems ought to be considered in their own right. Recently, other contributions have been developed in the literature by Fragnelli et al. (2014, 2015) trying to distribute indivisible units by mean of sequential algorithms.

Additionally, we assume that the society agrees on a list of social constraints; axioms delimiting a set of rules that society finds acceptable. The problem is that this set of rules may be too large. We devise a procedure that takes as given this list of social constraints and deduces a single-valued rule; for instance, Giménez-Gómez and Marco-Gil (2014) and Giménez-Gómez and Peris (2014a) analyse its application to the divisible case. Regarding this set of admissible rules, we define lower and upper bounds on awards by ensuring to each agent the smallest and the highest quantities they may receive from this set. Note that some actual indivisibility problems with bounds are present, such as the Spanish Parliament or the United States House of Representatives, where the seats are allocated to each province or state, respectively. Notwithstanding, a minimal number of seats is guaranteed to each province.

By applying our lower bound (upper bound) recursively on awards to distribute the endowment, we reach the discrete constrained equal losses (the discrete constrained equal awards) rule. These results are similar to those obtained by Herrero and Martínez (2008a), but our proposal guarantees order preservation and balancedness. Finally, we propose the simultaneous recursive application of both bounds on awards. That is, (i) each agent receives at least her lower bound, and (ii) her claim is bounded by her upper bound. This procedure ends up as the average of both of them.

The paper is organised as follows: Section 2 presents the model. Sections 3 and 4 provide our approaches and results. Finally, Section 5 concludes. Appendices gather technical proofs.

[1] See Thomson (2015) for a survey.

2. The model

Given the set of natural numbers $\mathbb{N}$, $\mathcal{N}$ denotes the class of non-empty finite subsets of $\mathbb{N}$. For a fixed set of agents $N = \{1, ..., n\} \in \mathcal{N}$, a **discrete claims problem** is a pair $(E, c) \in \mathbb{Z}_+ \times \mathbb{Z}_+^N$, where $\mathbb{Z}$ represents the set of integers, E denotes the endowment and $c = (c_1, \ldots, c_n)$ is the vector of claims for each agent $i \in N$, such that the aggregate claim is greater than the endowment, $C = \sum\limits_{i \in N} c_i \geq E$. Without loss of generality we assume claims are increasingly ordered, that is, for each $i < j$, $c_i \leq c_j$. The set of discrete claims problems is $\mathcal{D}$. An allocation x is admissible if it satisfies $x_i \geq 0$ (non-negativity), $x_i \leq c_i$ (claim-boundedness) and $\sum\limits_{i \in N} x_i = E$ (efficiency). Let $\mathcal{A}(E, c)$ be set of all admissible allocations for the problem (E, c), then the correspondence of admissible allocations is $Z : \mathcal{D} \rightrightarrows \bigcup_{(E,c) \in \mathcal{D}} \mathcal{A}(E, c)$. A **rule** φ is a single-valued selection from Z. Denote the set of rules by Φ. Each claims problem can be approached from two (dual) points of view: those of awards and losses, i.e. the part of the aggregate claims that are not honoured by the endowment ($L = C - E$); note that it will vary for each problem (E, c). The **dual** rule of φ (Aumann and Maschler, 1985), φ^d, assigns for each $(E, c) \in \mathcal{D}$ and each $i \in N$, $\varphi_i^d(E, c) = c_i - \varphi_i(L, c)$. Accordingly, a rule is **self-dual** (Aumann and Maschler, 1985) if it coincides with its dual.

Among all the rules proposed in the literature, we focus on the discrete constrained equal awards and the discrete constrained equal losses rule. In doing so, hereinafter, let σ denote a linear ordering (a complete, transitive and asymmetric binary relation) on the set of agents N. An agent i has higher priority over another agent j whenever $i\sigma j$. Ω denotes the set of all possible linear orderings on N. $Q(\varphi; E, c)$ represents the set of agents such that their assignment recommended by a rule (the first argument of Q) is not an integer. For any $x \in \mathbb{R}_+$, $\lfloor x \rfloor$ denotes the largest integer number s.t. $\lfloor x \rfloor \leq x$. Herrero and Martínez (2008a) propose that the discrete constrained equal awards rule associated with the ordering σ, $DCEA^\sigma$, recommends to each agent the integer part of the assignment given by the CEA rule, and the remaining endowment is distributed following a priority ordering σ among the agents in $Q(CEA; E, c)$. According to the priority order, we give one unit to each of the claimants in $Q(CEA; E, c)$ with the highest priority until what remains of the endowment is distributed.[2]
Given $\sigma \in \Omega$, for each $(E, c) \in \mathcal{D}$ and each $i \in N$, the **discrete constrained equal awards** rule associated with σ, $DCEA^\sigma$ is: $DCEA_i^\sigma(E, c) = \lfloor CEA_i(E, c) \rfloor + 1$ if i is in the list of the E' elements with highest priority in $Q(CEA; E, c)$, where $E' = E - \sum\limits_{i \in N} \lfloor CEA_i(E, c) \rfloor > 0$; or $\lfloor CEA_i(E, c) \rfloor$, otherwise.

Similarly, the discrete constrained equal losses rule associated with the ordering σ, $DCEL^\sigma$, recommends to each agent the integer part of CEL rule,[3] and the remaining endowment is distributed following a priority ordering σ among the agents in $Q(CEL; E, c)$, until what remains of the endowment is distributed.
Given $\sigma \in \Omega$, for each $(E, c) \in \mathcal{D}$ and each $i \in N$, the **discrete constrained equal losses**

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[2]Let $\mathcal{D}$ denote the set of perfectly divisible claims problems. The **constrained equal awards** rule (Maimonides 12th, among others), CEA: for each $(E, c) \in \mathcal{D}$ and each $i \in N$, $CEA_i(E, c) \equiv \min\{c_i, \mu\}$, where μ is chosen so that $\sum\limits_{i \in N} \min\{c_i, \mu\} = E$.

[3]The **constrained equal losses** rule (Aumann and Maschler, 1985), CEL: for each $(E, c) \in \mathcal{D}$ and each $i \in N$, $CEL_i(E, c) \equiv \max\{0, c_i - \mu\}$, where μ is chosen so that $\sum\limits_{i \in N} \max\{0, c_i - \mu\} = E$.

rule associated with σ, $DCEL^\sigma$ is: $DCEL_i^\sigma(E,c) = \lfloor CEL_i(E,c) \rfloor + 1$ if i is in the list of E' elements with highest priority in $Q(CEL; E, c)$, where $E' = E - \sum_{i \in N} \lfloor CEL_i(E,c) \rfloor > 0$; or $\lfloor CEL_i(E,c) \rfloor$, otherwise.

3. The single recursive process

It is noteworthy that in many actual rationing situations the admissible allocations are determined by legal requirements or socially accepted axioms. The class of axioms is a set $\mathcal{A}$. Accordingly, we propose to delimit the sets of admissible allocations in terms of some **list of axioms** P. Let $\Phi[P]$ be the set of rules satisfying all axioms in P, $\Phi[P] \subset \Phi \subset (\mathcal{R}^N)^{\mathcal{D}}$. Therefore, for each $(E,c) \in \mathcal{D}$, each agent should receive at least the smallest amount recommended to her by all the proposals in $\Phi[P]$ — **the P-rights** value Pr.

The aggregate guaranteed amount by means of the Pr may not exhaust the endowment, so a requirement of composition from the profile of these bounds arises in a natural way. In this sense, we follow Giménez-Gómez and Marco-Gil (2014) in order to distribute the endowment through the application of this lower bound. Specifically, we define the single recursive process such that at the first step each agent will receive her Pr of the original problem. At the second step, the endowment is what remains and the claims are adjusted down by the amounts just assigned. Then, each agent receives her Pr in this residual problem, and so on.

Definition 1 *Given a list of axioms $P \in 2^{\mathcal{A}}$, for each $(E,c) \in \mathcal{D}$, each $i \in N$ and each step $m \in \mathbb{N}$, $Pr_i(E^m, c^m)$ is defined in the following way:*
Step 1, $(E^1, c^1) = (E, c)$ and $Pr_i(E^1, c^1) = \min_{\varphi \in \Phi[P]} \{\varphi_i(E^1, c^1)\}$.
Step m, for $m \geq 2$, $(E^m, c^m) \equiv (E^{m-1} - \sum_{i \in N} Pr_i(E^{m-1}, c^{m-1}), c^{m-1} - Pr(E^{m-1}, c^{m-1}))$, and $Pr_i(E^m, c^m) = \min_{\varphi \in \Phi[P]} \{\varphi_i(E^m, c^m)\}$. and so on untill $m = t$ where $E^t = 0$.
Given $\sigma \in \Omega$, the ***single recursive process*** *defines the mapping $R^\sigma : 2^{\mathcal{A}} \to \Phi$ and $R^\sigma[P]_i(E,c)$ $= \sum_{m=1}^{t} Pr_i(E^m, c^m)$+1, if i is in the list of the E' elements with highest priority, where $E' = E - \sum_{i \in N} \sum_{m=1}^{t} Pr_i(E^m, c^m) > 0$; $R^\sigma[P]_i(E,c) = \sum_{m=1}^{t} Pr_i(E^m, c^m)$, otherwise.*

Note that this kind of process has already been used in the context of conflicting claims problems by Alcalde et al. (2005), Dominguez and Thomson (2006), and Dominguez (2013), among others. Additionally, the result of this single recursive process depends on P and on the priority ordering σ. Therefore, it is straightforward to see that this process exhausts the endowment, since if each agent's Pr value is not positive, then priority ordering, which is exogenous, is applied at the end of the process on the set of allocations that satisfy all the properties in P to distribute the last remaining units in order to reach efficiency.

We consider two axioms that have been used by many authors as a minimal requirement of fairness (see, for instance, Thomson, 2015): order preservation and balancedness. The former axiom requires respecting the ordering of the claims, i.e. if agent $i's$ claim is at least as large as agent $j's$ claim, she should receive and lose at least as much as agent j does, respectively. Regarding balancedness, note that one of the most traditional fairness properties is to treat identical agents identically. It simply means that agents with identical claims should receive the same amount. This requirement cannot be unconditionally met in the indivisibilities case, hence we relax this property by simply asking identical agents to be "almost" treated equally in terms of their awards. More precisely, equal agents get amounts that can differ only in one unit, which represents the size of the indivisibility.

Order preservation, OP**:** For each $(E, c) \in \mathcal{D}$ and each pair $i, j \in N$, if $c_i > c_j$, then $\varphi_i(E, c) \geq \varphi_j(E, c)$ and $c_i - \varphi_i(E, c) \geq c_j - \varphi_j(E, c)$.

Balancedness, BAL**:** For each $(E, c) \in \mathcal{D}$ and each pair $i, j \in N$, if $c_i = c_j$, then $\mid \varphi_i(E, c) - \varphi_j(E, c) \mid \leq 1$.

The non-fulfilment of these axioms allows that, in a disaster situation, people who need more assistance may receive less than those who need less; or if an airport has some slots available and there are two identical airline companies that are demanding the same number of airport slots, then one company could obtain all of the slots. In this sense, the following example shows that in the M-down method, depending on which priority order we apply, order preservation may not fulfil.

Example 1 *Consider three patients that are waiting for a corneal transplant. The first two patients need one cornea, whereas the third one needs two. There are two corneas available. Hence, their associated claims problem is* $(E, c) = (2, (1, 1, 2))$*. In this case,* $CEA(E, c) = (\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$*, and* $CEL(E, c) = (\frac{1}{3}, \frac{1}{3}, 1\frac{1}{3})$*. Then,* $\lfloor CEA(E, c) \rfloor = (0, 0, 0)$*, and* $\lfloor CEL(E, c) \rfloor = (0, 0, 1)$*. Consider that the priority order* $\sigma \in \Omega$*, that is determined by the time that each patient has been waiting the transplant, is* $1\sigma 2\sigma 3$*. Hence, by the M-down method* $DCEA^{\sigma}(E, c) = (1, 1, 0)$ *and* $DCEL^{\sigma}(E, c) = (1, 0, 1)$*, contradicting OP.*

Thereafter, requiring the fulfilment of the accepted axioms P appears in a natural way. Particularly, given an ordering σ, the discrete constrained equal awards and the discrete constrained equal loses rules with principles, $DCEA^{\sigma}[P]$ and $DCEL^{\sigma}[P]$, respectively, by means of the order σ selects an allocation that satisfyes P. Formally,

Definition 2 *Given a list of axioms* $P \in 2^{\mathcal{A}}$ *and* $\sigma \in \Omega$*, for each* $(E, c) \in \mathcal{D}$ *and each* $i \in N$*, the* ***discrete constrained equal awards rule with principles associated to*** σ*,* $DCEA^{\sigma}[P]$*, is:* $DCEA^{\sigma}_i[P](E, c) = \lfloor CEA_i(E, c) \rfloor + 1$ *if* i *is in the list of the* E' *elements in* $Q(CEA; E, c)$ *with highest priority in the order* σ*; or* $\lfloor CEA_i(E, c) \rfloor$*, otherwise, whenever* $DCEA^{\sigma}[P](E, c)$ *satisfies all principles in* P*. If not, we repeat the same process with the following priority order in* σ *until* $DCEA^{\sigma}[P](E, c)$ *satisfies all principles in* P*.* [4]

Definition 3 *Given a list of axioms* $P \in 2^{\mathcal{A}}$ *and* $\sigma \in \Omega$*, for each* $(E, c) \in \mathcal{D}$ *and each* $i \in N$*, the* ***discrete constrained equal loses rule with principles associated to*** σ*,* $DCEL^{\sigma}[P]$*, is:* $DCEL^{\sigma}_i[P](E, c) = \lfloor CEL_i(E, c) \rfloor + 1$ *if* i *is in the list of the* E' *elements in* $Q(CEL; E, c)$ *with highest priority in the order* σ*; or* $\lfloor CEL_i(E, c) \rfloor$*, otherwise, whenever* $DCEL^{\sigma}[P](E, c)$ *satisfies all principles in* P*. If not, we repeat the same process with the following priority order in* σ *until* $DCEL^{\sigma}[P](E, c)$ *satisfies all principles in* P*.* [5]

[4]Recall that $E' = E - \sum\limits_{i \in N} \lfloor CEA_i(E, c) \rfloor > 0$.

[5]Recall that $E' = E - \sum\limits_{i \in N} \lfloor CEL_i(E, c) \rfloor > 0$.

Hence, for instance, the $DCEL^\sigma[P]$ works as follows. Firstly, recommends to each agent the integer part of the assignment given by the CEL rule. Secondly, the remaining endowment is distributed among the agents in $Q(CEL; E, c)$ according to σ, if the obtained allocation satisfies all principles in P, this is the $DCEL^\sigma[P]$. Otherwise, we distribute again the remaining endowment with the following priority order in σ and so on until we get an allocation satisfying all the required principles in P.

Example 2 *(continues from Example 1) Consider $P = \{OP, BAL\}$. In this case, $\lfloor CEA(E,c) \rfloor = (0,0,0)$. Consider the priority order $1\sigma2\sigma3$. We have two remaining units to distribute, according to the priority order the distribution should be $(1,1,0)$ but this allocation is not admissible since fails OP. Then, we apply the following priority order and the allocation we get is $(1,0,1)$, since this satisfies OP and BAL we have $DCEA^\sigma[P](E,c) = (1,0,1)$. Similarly, the $\lfloor CEL(E,c) \rfloor = (0,0,1)$ and we have one remaining unit to distribute. According to the priority order the distribution should be $(1,0,1)$ and this satisfies all principles P, therefore this is the $DCEL^\sigma[P](E,c)$.*

The next result states, as Example 3 shows, that under OP and BAL, the $DCEL^\sigma[P]$ is obtained by $R^\sigma[P]$.

Theorem 1 *Given $\sigma \in \Omega$ and $P = \{OP, BAL\}$, for each $(E,c) \in \mathcal{D}$, $R^\sigma[P](E,c) = DCEL^\sigma[P](E,c)$.*

Proof. See Appendix 1.

Note that this result states that the recursive application of a lower bound that may be easily accepted ends up to a distribution of the endowment that favours highest claimants.

Example 3 *Consider $(E,c) = (100,(20,30,60))$, $P = \{OP, BAL\}$ and the priority ordering $\sigma \in \Omega$ is $2\sigma3\sigma1$. At step $m = 1$, $CEA(E^1,c^1) = (20,30,50) = DCEA^\sigma[P](E^1,c^1)$ and $CEL(E^1,c^1) = (16\frac{2}{3}, 26\frac{2}{3}, 56\frac{2}{3})$. Then $\lfloor CEL(E^1,c^1) \rfloor = (16,26,56)$, so there are two remaining units to distribute among the three agents, giving three plausible allocations $(17,27,56)$, $(17,26,57)$ and $(16,27,57)$. By P, the only admissible one is $(17,27,56)$. Thus, $DCEL^\sigma[P](E^1,c^1) = (17,27,56)$.*

By Giménez-Gómez and Marco-Gil (2014), when considering OP and BAL, the CEL and CEA rules are the Pr for agents 1 and n, respectively. For agents in between we must look for the smallest allocation satisfying both properties. Therefore, in our case, the admissible allocation that provides the smallest amount to agent 2 is $(20,25,55)$. Hence, $Pr(E^1,c^1) = (17,25,50)$.

At step $m = 2$, $(E^2,c^2) = (8,(3,5,10))$. So that, $CEA(E^2,c^2) = (2\frac{2}{3}, 2\frac{2}{3}, 2\frac{2}{3})$, then $\lfloor CEA(E^2,c^2) \rfloor = (2,2,2)$, but there are two more units to distribute and this gives three possible allocations $(2,3,3)$, $(3,2,3)$, and $(3,3,2)$, by P the only admissible one is $(2,3,3)$, thus $DCEA^\sigma[P](E^2,c^2) = (2,3,3)$.

The $CEL(E^2,c^2) = (0, 1\frac{1}{2}, 6\frac{1}{2})$ and $\lfloor CEA(E^2,c^2) \rfloor = (0,1,6)$, there is one more unit to distribute among agents 2 and 3. This gives two possibilities $(0,2,6)$ and $(0,1,7)$, by P the only admissible one is $(0,2,6)$. Thus, $DCEL^\sigma[P](E^2,c^2) = (0,2,6)$. Furthermore, note that there is no other admissible allocation that recommends a smaller amount to agent 2. Hence $Pr(E^2,c^2) = (0,2,3)$.

At step $m = 3$*,* $(E^3, c^3) = (3, (3, 3, 7))$*. So that,* $CEA(E^3, c^3){=}(1, 1, 1)$ *and* $CEL(E^3, c^3)$ $= (0, 0, 3)$*. So,* $DCEA^\sigma[P](E^3, c^3) = (1, 1, 1)$*,* $DCEL^\sigma[P](E^3, c^3) = (0, 0, 3)$*, and* $Pr(E^3, c^3)$ $= (0, 0, 1)$*. From* $m = 3$ *on, agents* 1 *and* 2 *will not receive any awards and all the remaining units will go for the third agent. Therefore,* $R^\sigma[P](E, c) = (17, 27, 56) = DCEL^\sigma[P](E, c)$*.*

Since $DCEL$ and $DCEA$ are dual to each other (Herrero and Martínez, 2004), the next corollary is straightforward.

Corollary 1 *Given* $\sigma \in \Omega$ *and* $P = \{OP, BAL\}$*, for each* $(E, c) \in \mathcal{D}$*,* $c - R^\sigma[P](L, c) = DCEA^\sigma[P](E, c)$*.*

4. The double recursive process

Next, our notion of an admissible rule is given by both P and an admissible zone delimited by two dual reference rules $F[P]$ and $F[P]^d$. An admissible rule $\varphi \in \Phi[P]$ is **PF-admissible** if $\min\{F[P]_i(E, c), F[P]^d_i(E, c)\} \leq \varphi_i\,(E, c) \leq \max\{F[P]_i(E, c), F[P]^d_i(E, c)\}$. Let $\Phi[P; F]$ denote the set of PF-admissible rules.

Note that this framework fits with conflict situations, such as divorce, where a neutral third agent (the mediator) decides the allocation that should be applied in accordance with some legitimate axioms (the law) and two prominent points of view. Moreover, since in claims problems we have two points of view (those of awards and losses), the two dual reference rules arise naturally.[6] Furthermore, since since F and F^d satisfy P, then P will be a set of self-dual axioms.

The idea of ensuring to each agent her Pr value may be understood as giving her the part of her claims that is not disputed. It is also commonly assumed that the part of the claims that is not feasible should not be considered.[7] Thus, we truncate the claims by the upper bound given by each agent's **P-utopia** value Pu. Therefore, following Giménez-Gómez and Peris (2014a), we propose the double recursive process in which, at any step, each agent receives her Pr value and, at the same time, her claim is truncated by her Pu value.

Definition 4 *Given a list of self-dual axioms* $P \in 2^{\mathcal{A}}$*, for each* $(E, c) \in \mathcal{D}$*, each* $i \in N$ *and each step* $m \in \mathbb{N}$*,* $Pr_i(E^m, c^m)$ *and* $Pu_i(E^m, c^m)$ *are defined in the following way:*
Step 1*, for* $m = 1$*,*

$$(E^1, c^1) = (E, c),\ Pr_i(E^1, c^1) = \min_{\varphi\in\Phi[P;F]} \{\varphi_i(E^1, c^1)\},\ \text{and}$$
$$Pu_i(E^1, c^1) = \max_{\varphi\in\Phi[P;F]} \{\varphi_i(E^1, c^1)\}.$$

Step m*, for* $m \geq 2$*,*

$$E^m = E^{m-1} - \sum_{i\in N} Pr_i(E^{m-1}, c^{m-1}),\ c^m_i = Pu_i(E^{m-1}, c^{m-1}) - Pr_i(E^{m-1}, c^{m-1}),$$
$$Pr_i(E^m, c^m) = \min_{\varphi\in\Phi[P;F]} \{\varphi_i(E^m, c^m)\}\ \text{and}$$
$$Pu_i(E^m, c^m) = \max_{\varphi\in\Phi[P;F]} \{\varphi_i(E^m, c^m)\}.$$

Given $\sigma \in \Omega$*, the* ***double recursive process*** *defines the mapping* $DR^\sigma : 2^{\mathcal{A}} \to \Phi$ *and* $DR^\sigma[P; F]_i(E, c) = \sum_{m=1}^{\infty} Pr_i(E^m, c^m)$*+1, if* i *is in the list of the* E' *elements with highest*

[6] Giménez-Gómez and Peris (2014a) discuss mediation situations.

[7] See, for instance, Aumann and Maschler (1985).

priority, where $E' = E - \sum_{i\in N}\sum_{m=1}^{\infty} Pr_i(E^m, c^m) > 0$*;* $DR^\sigma[P;F]_i(E,c) = \sum_{m=1}^{\infty} Pr_i(E^m, c^m)$*, otherwise.*

This process always satisfies efficiency due to the application of σ, as in the case of the recursive process. Furthermore, DR^σ ends at the midpoint of the allocations provided by the P-rights and the P-utopia (similarly, at the average of F and F^d).

Theorem 2 *Given* $\sigma \in \Omega$*, and* $P \in 2^{\mathcal{A}}$*, for each* $(E,c) \in \mathcal{D}$ *and each* $i \in N$*,* $DR^\sigma[P;F]_i(E,c) = \lfloor \frac{Pr_i(E,c)+Pu_i(E,c)}{2} \rfloor + 1 = \lfloor \frac{F[P]_i(E,c)+F[P]_i^d(E,c)}{2} \rfloor + 1$*, if* i *is in the list of* E' *elements with highest priority in* $Q(\frac{Pr(E,c)+Pu(E,c)}{2}; E, c)$*, where* $E' = E - \sum_{i\in N} \lfloor \frac{Pr_i(E,c)+Pu_i(E,c)}{2} \rfloor > 0$ *or* $\lfloor \frac{Pr_i(E,c)+Pu_i(E,c)}{2} \rfloor = \lfloor \frac{F[P]_i(E,c)+F[P]_i^d(E,c)}{2} \rfloor$*, otherwise.*

Proof. See Appendix 2.

Example 4 *Consider* $(E,c) = (100,(20,30,60))$*,* $P = \{OP, BAL\}$*, the priority ordering* $\sigma \in \Omega$ *is* $2\sigma3\sigma1$*, and* $F[P](E,c) = DCEA^\sigma[P]$*,* $F[P]^d(E,c) = DCEL^\sigma[P]$*. At step* $m = 1$*, from Example 3,* $DCEA^\sigma[P](E^1,c^1) = (20,30,50)$ *and* $DCEL^\sigma[P](E^1,c^1) = (17,27,56)$*. Hence,* $Pr(E^1,c^1) = (17,27,50)$ *and* $Pu(E^1,c^1) = (20,30,56)$*. At step* $m = 2$*,* (E^2,c^2)=$(6,(3,3,6))$*. So that,* $CEA(E^2,c^2) = (2,2,2)$ *and* $CEL(E^2,c^2) = (1,1,4)$*. Then,* $DCEA^\sigma[P](E^2,c^2) = (2,2,2)$ *and* $DCEL^\sigma[P](E^2,c^2) = (1,1,4)$*. Hence,* $Pr(E^2,c^2) = (1,1,2)$*, and* $Pu(E^2,c^2) = (2,2,4)$*. At step* $m = 3$*,* (E^3,c^3)=$(2,(1,1,2))$*. So that,* $CEA(E^3,c^3) = (\frac{2}{3},\frac{2}{3},\frac{2}{3})$ *and* $CEL(E^3,c^3) = (\frac{1}{3},\frac{1}{3},1\frac{1}{3})$*. Then,* $DCEA^\sigma[P](E^3,c^3) = (0,1,1)$ *and* $DCEL^\sigma[P](E^3,c^3) = (0,1,1)$*.* [8] *So,* $Pr(E^3,c^3) = Pu(E^3,c^3) = (0,1,1)$*. Therefore,* $DR^\sigma[P;F](E,c) = (18,29,53)$*. Additionally,* $\frac{DCEA^\sigma[P](E,c)+DCEL^\sigma[P](E,c)}{2} = \frac{Pr(E,c)+Pu(E,c)}{2} = (18\frac{1}{2},28\frac{1}{2},53)$*, so* $\lfloor \frac{DCEA^\sigma[P](E,c)+DCEL^\sigma[P](E,c)}{2} \rfloor = (18,28,53)$ *and the remaining unit corresponds to player* 2*, according to* σ*, i.e.,* $(18,29,53)$*, which coincides with* $DR^\sigma[P;F]$*.*

5. Final remarks

The current approach has proposed two procedures for allocating a discrete endowment. Specifically, besides the usual priority ordering σ, a list of axioms P has been taken into account. In this manner, it is ensured that in situations where the claimants are prioritised following the arrival ordering, if at any time a change in the claims occurs, P determines the admissible allocations. For instance, consider medical assistance where $P = \{OP, BAL\}$. If, suddenly, there is a natural disaster causing the biggest demand of medical assistance by a country that has suffered it last, then despite being the last to demand medical assistance, by the requirement of P this country will receive at least as much as the previous countries.

[8] $DCEA^\sigma(E^3,c^3,P)$ may recommend $(1,1,0)$, $(1,0,1)$ and $(0,1,1)$, but only $(1,0,1)$ and $(0,1,1)$ are PF-admissible. Then, by σ, $DCEA^\sigma[P](E^3,c^3) = (0,1,1)$. Moreover, $DCEL^\sigma(E^3,c^3,P)$ may recommend $(0,1,1)$, $(1,0,1)$ and $(0,0,2)$, but only $(1,0,1)$ and $(0,1,1)$ are PF-admissible. Then, by σ, $DCEL^\sigma[P](E^3,c^3)$ $= (0,1,1)$.

Appendix 1 *Proof of Theorem 1*

We assume throughout Appendix 1 that $P = \{OP, BAL\}$. The proof is based on five lemmas, two remarks and one fact.

Remark 1 *Given P, for each $(E, c) \in \mathcal{D}$ and each $i, j \in N$, if $c_i^m \leq c_j^m$, then $c_i^{m+1} \leq c_j^{m+1}$.*

Proof.

Let $(E, c) \in \mathcal{D}$, $i, j \in N$ such that $c_i^m \leq c_j^m$ and φ^*, φ' belonging to $\Phi[P]$. By OP, for each $\varphi \in \Phi[P]$, $c_i^m - \varphi_i(E^m, c^m) \leq c_j^m - \varphi_j(E^m, c^m)$ so that,

(a) If $Pr_i^m(E, c) = \varphi_i^*(E^m, c^m)$ and $Pr_j^m(E, c) = \varphi_j^*(E^m, c^m)$, by OP, $c_i^m - Pr_i^m(E^m, c^m) \leq c_j^m - Pr_j^m(E^m, c^m)$. Therefore, $c_i^{m+1} \leq c_j^{m+1}$.

(b) If $Pr_i^m(E, c) = \varphi_i^*(E^m, c^m)$ and $Pr_j^m(E, c) = \varphi_j'(E^m, c^m)$, by construction, $\varphi_j'(E^m, c^m) \leq \varphi_j^*(E^m, c^m)$, so that, $c_i^m - \varphi_i^*(E^m, c^m) \leq c_j^m - \varphi_j^*(E^m, c^m) \leq c_j^m - \varphi_j'(E^m, c^m)$. Therefore, $c_i^{m+1} \leq c_j^{m+1}$.

$\blacksquare$

Fact 1 *For each $(E, c) \in \mathcal{D}$ and each $i \in N$, taking into the account that the loss imposed on agent i by CEL is $CEL_i(E, c) = c_i - \gamma_i$, where $\gamma_i = \min\{c_i, \alpha_i\}$ and $\alpha_i = (L - \sum_{j<i} \gamma_j) / (n - i + 1)$; and, $L^m = \sum_{i \in N} \left(c_i - \sum_{k=1}^{m} Pr_i(E^k, c^k)\right) - \left(E - \sum_{i \in N} \sum_{k=1}^{m} Pr_i(E^k, c^k)\right) = C - E = L$, we obtain that:*

(a) If $\gamma_i = c_i$, then for each $j < i$, $\gamma_j = c_j$.

(b) If $\gamma_i = \alpha_i$, then $\alpha_i = \mu$, and for each $j > i$, $\alpha_j = \alpha_i$. Therefore, $\gamma_i = \mu$.

(c) At each $m \in \mathbb{N}$ and for each $i \in N$, α_i^m only depends on the initial problem, (E, c), and on agent j's claim, for each $j < i$.[9]

Remark 2 *Given $\sigma \in \Omega$ and P, for each $(E, c) \in \mathcal{D}$ and each $i \in N$, $DCEL^\sigma[P]_i(E, c) = c_i - \min\{c_i, \bar{\mu}\}$, where $\bar{\mu} = \tilde{\mu} - 1$, if i is one of the E' elements with highest priority ordering in $Q(CEL; E, c)$; or $\bar{\mu} = \tilde{\mu}$, otherwise; and $\tilde{\mu} = \mu$, if $\mu \in \mathbb{Z}$; or $\tilde{\mu} = \lfloor \mu \rfloor + 1$, if $\mu \notin \mathbb{Z}$. If $c_i - \min\{c_i, \bar{\mu}\}$ satisfies all axioms in P. If not, we repeat the same process with the following priority ordering in σ until all axioms in P are satisfied.*

Proof. *For each $i \in N$, by definition $\lfloor CEL_i(E, c) \rfloor = \lfloor c_i - \min\{c_i, \mu\} \rfloor$. We distinguish two cases:*

***Case 1**: $\min\{c_i, \mu\} = c_i$. Since $\tilde{\mu} \geq \mu$ then, $\lfloor CEL_i(E, c) \rfloor = \lfloor c_i - \min\{c_i, \mu\} \rfloor = c_i - \min\{c_i, \tilde{\mu}\} = 0$.*

***Case 2**: $\min\{c_i, \mu\} = \mu$. We have two possibilities:*

2.1 *If $\mu \in \mathbb{Z}$, then $\tilde{\mu} = \mu \in \mathbb{Z}$ and $\lfloor CEL_i(E, c) \rfloor = \lfloor c_i - \min\{c_i, \mu\} \rfloor = \lfloor c_i - \mu \rfloor = c_i - \mu = c_i - \tilde{\mu} = c_i - \min\{c_i, \tilde{\mu}\}$.*

2.2 *If $\mu \notin \mathbb{Z}$, then $\tilde{\mu} = \lfloor \mu \rfloor + 1 > \mu$ and $\lfloor CEL_i(E, c) \rfloor = \lfloor c_i - \min\{c_i, \mu\} \rfloor = \lfloor c_i - \mu \rfloor$. Since $\mu \notin \mathbb{Z}$ we have $\lfloor \mu \rfloor < \mu < \lfloor \mu \rfloor + 1$ and $\mu = \lfloor \mu \rfloor + \xi$ where $\xi \in (0, 1)$. Thus, $c_i - (\lfloor \mu \rfloor + 1) < c_i - \mu < c_i - \lfloor \mu \rfloor$ and $\lfloor c_i - \mu \rfloor = \lfloor c_i - (\lfloor \mu \rfloor + \xi) \rfloor = c_i - (\lfloor \mu \rfloor + 1) = c_i - \tilde{\mu}$. Moreover, $c_i > \mu$*

[9] α_i^m denotes the α of agent i at step m.

and $\mu \notin \mathbb{Z}$ so, we have $c_i \geq \lfloor \mu \rfloor + 1 = \tilde{\mu}$. Thus, $c_i - (\lfloor \mu \rfloor + 1) = c_i - \tilde{\mu} = c_i - \min\{c_i, \tilde{\mu}\}$. Therefore, $\lfloor CEL_i(E,c) \rfloor = c_i - \min\{c_i, \tilde{\mu}\}$. Hence, $DCEL^\sigma[P]_i(E,c) = c_i - \min\{c_i, \bar{\mu}\}$.

■

Lemma 1 *Given $\sigma \in \Omega$ and P, for each $(E,c) \in \mathcal{D}$, each $i \in N$ and each $m \in \mathbb{N}$, $\bar{\mu}^{m+1} = \bar{\mu}^m$ where $\bar{\mu}^m$ solves $\sum_{i \in N} \lfloor CEL_i(E^m, c^m) \rfloor = E^m$ and $\bar{\mu}^{m+1}$ solves $\sum_{i \in N} \lfloor CEL_i(E^{m+1}, c^{m+1}) \rfloor = E^{m+1}$.*

Proof. *Let agent $i \in N$ be the first agent who receives a positive amount at step $m \in \mathbb{N}$ according to the $DCEL^\sigma$ rule, i.e. (i) $DCEL_i^\sigma(E^m, c^m) > 0$ and (ii) for each $j \in N : j < i$, $DCEL_j^\sigma(E^m, c^m) = 0$. By (i) and Fact 1, $c_i^m > \bar{\mu}^m = \alpha_i^m$. Given (ii) and Definition 1 at the m-th step, $c_j^{m+1} = c_j^m$. By Fact 1(c), $\alpha_i^{m+1} = \alpha_i^m = \bar{\mu}^m < c_i^m$. Furthermore, $c_i^{m+1} = c_i^m - \min_{\varphi \in \Phi[P]} \{\varphi_i(E^m, c^m)\} \geq c_i^m - DCEL^\sigma{}_i(E^m, c^m) = c_i^m - (c_i^m - \bar{\mu}^m) = \bar{\mu}^m = \alpha_i^{m+1}$. Therefore, by Fact 1, $\gamma_i^{m+1} = \alpha_i^{m+1} = \bar{\mu}^{m+1}$.*

From now on, μ and $\bar{\mu}$ denote μ^m and $\bar{\mu}^m$, respectively, for each $m \in \mathbb{N}$.

Lemma 2 *Given $\sigma \in \Omega$ and P, for each $(E,c) \in \mathcal{D}$ if there is $m \in \mathbb{N}$ such that $Pr_i(E^m, c^m) = DCEL^\sigma[P]_i(E^m, c^m)$. Then, for each $h \in \mathbb{N}$, $Pr_i(E^{m+h}, c^{m+h}) = 0$.*

Proof. *For each $i \in N$, we show that if $Pr_i(E^m, c^m) = DCEL^\sigma[P]_i(E^m, c^m)$ then $Pr_i(E^{m+1}, c^{m+1}) = DCEL^\sigma[P]_i(E^{m+1}, c^{m+1}) = 0$. Let $(E,c) \in \mathcal{D}$ and $m \in \mathbb{N}$, be such that $Pr_i(E^m, c^m) = DCEL^\sigma[P]_i(E^m, c^m) = c_i^m - \min\{c_i^m, \bar{\mu}\}$. Then, ${c_i}^{m+1} = {c_i}^m - DCEL^\sigma[P]_i(E^m, c^m) = {c_i}^m - (c_i^m - \min\{c_i^m, \bar{\mu}\}) = \min\{c_i^m, \bar{\mu}\}$. Thus, $DCEL^\sigma[P]_i(E^{m+1}, c^{m+1}) = c_i^{m+1} - \min\left\{c_i^{m+1}, \bar{\mu}\right\} = 0$. Since for each $(E,c) \in \mathcal{D}$ and each $i \in N$, the loss imposed on agent i by CEL is $CEL_i(E,c) = c_i - \gamma_i$, where $\gamma_i = \min\{c_i, \alpha_i\}$ and $\alpha_i = (L - \sum_{j<i} \gamma_j) / (n - i + 1)$ if $DCEL^\sigma[P]_i(E^{m+1}, c^{m+1}) = 0$ then, for each $h \in \mathbb{N}$, $Pr_i(E^{m+1}, c^{m+1}) = DCEL^\sigma[P]_i(E^{m+h}, c^{m+h}) = 0$.*

■

Lemma 3 *Given $\sigma \in \Omega$ and P, for each $(E,c) \in \mathcal{D}$ and each $i \in N$, if for each $m \in \mathbb{N}$, $Pr_i(E^m, c^m) = \varphi_i(E^m, c^m) \neq DCEL^\sigma[P]_i(E,c)$, $\sum_{k=1}^{\infty} Pr_i(E^k, c^k) \leq DCEL^\sigma[P]_i(E,c)$.*

Proof. *Suppose that for each $m \in \mathbb{N}$ and each $i \in N$, $Pr_i(E^m, c^m) = \varphi_i(E^m, c^m) \neq DCEL^\sigma[P]_i(E,c)$. By Remark 2, for each $m \in \mathbb{N}$, $DCEL^\sigma[P]_i(E^m, c^m) = c_i^m - \min\{c_i^m, \bar{\mu}\}$, $Pr_i(E^m, c^m) \leq DCEL^\sigma[P]_i(E^m, c^m) = \lfloor c_i^m - \mu \rfloor = c_i^m - \bar{\mu} = c_i - \sum_{k=1}^{m-1} Pr_i(E^k, c^k) - \bar{\mu}$. Thus, $Pr_i(E^m, c^m) + \sum_{k=1}^{m-1} Pr_i(E^k, c^k) \leq c_i - \bar{\mu} = DCEL^\sigma[P]_i(E,c)$, that is $\sum_{k=1}^{m} Pr_i(E^k, c^k) \leq DCEL^\sigma[P]_i(E,c)$. Therefore, $\lim_{m \to \infty} \sum_{k=1}^{m} Pr_i(E^k, c^k) \leq DCEL^\sigma[P]_i(E,c)$.*

■

Lemma 4 *Given $\sigma \in \Omega$ and P, for each $(E,c) \in \mathcal{D}$, and each $i \in N$, if there is $m^* \in \mathbb{N}$, $m^* > 1$, such that $Pr_i(E^{m^*}, c^{m^*}) = DCEL^\sigma[P]_i(E^{m^*}, c^{m^*})$ and $Pr_i(E^{m^*-1}, c^{m^*-1}) =$*

$\varphi_i(E^{m^*-1}, c^{m^*-1}) \neq DCEL^\sigma[P]_i(E^{m^*-1}, c^{m^*-1})$, *then* $\sum_{k=1}^{m^*} Pr_i(E^k, c^k) = DCEL^\sigma[P]_i(E, c)$.

Proof. *Let* $m^* \in \mathbb{N}$, $m^* > 1$ *be such that, for each* $i \in N$, $Pr_i(E^{m^*}, c^{m^*}) = DCEL^\sigma[P]_i(E^{m^*}, c^{m^*})$ *and* $Pr_i(E^{m^*-1}, c^{m^*-1}) = \varphi_i(E^{m^*-1}, c^{m^*-1}) \neq DCEL^\sigma[P]_i(E^{m^*-1}, c^{m^*-1})$. *Since,* $\varphi_i(E^{m^*-1}, c^{m^*-1}) < DCEL^\sigma[P]_i(E^{m^*-1}, c^{m^*-1})$, $DCEL^\sigma[P]_i(E^{m^*-1}, c^{m^*-1},) > 0$. *By Lemma 2,* $DCEL^\sigma[P]_i(E^{m^*}, c^{m^*}) = c_i^{m^*} - \min\{c_i^{m^*}, \bar{\mu}\}$. *Since,* $DCEL^\sigma[P]_i(E^{m^*-1}, c^{m^*-1}) > 0$, *then* $c_i^{m^*-1} > \bar{\mu}$. *By Lemma 1,* $c_i^{m^*} \geq \bar{\mu}$. *Then, at step* m^*, *agent* i *has received* $\sum_{k=1}^{m^*} Pr_i(E^k, c^k) = \sum_{k=1}^{m^*-1} Pr_i(E^k, c^k) + DCEL^\sigma[P]_i(E^{m^*}, c^{m^*}) = \sum_{k=1}^{m^*-1} Pr_i(E^k, c^k) + (c_i^{m^*} - \min\{c_i^{m^*}, \bar{\mu}\}) = \sum_{k=1}^{m^*-1} Pr_i(E^k, c^k) + c_i - \sum_{k=1}^{m^*-1} Pr_i(E^k, c^k) - \min\{c_i^{m^*}, \bar{\mu}\} = c_i - \min\{c_i^{m^*}, \bar{\mu}\} = c_i - \bar{\mu} = DCEL^\sigma[P]_i(E, c)$.

∎

Lemma 5 *Given* $\sigma \in \Omega$ *and* P, *for each* $(E, c) \in \mathcal{D}$, $Pr_1(E, c) = DCEL^\sigma[P]_1(E, c)$ *and* $Pr_n(E, c) = DCEA^\sigma[P]_n(E, c)$.

Proof. *We show that* $Pr_1(E, c) = DCEL^\sigma[P]_1(E, c)$. *Consider the two following cases:*
- $DCEL^\sigma[P]_1(E, c) = 0$. *By non-negativity,* $Pr_1(E, c) = DCEL^\sigma[P]_1(E, c)$.
- $DCEL^\sigma[P]_1(E, c) > 0$. *By the* $DCEL^\sigma[P]$ *rule definition and* OP, $c_1 - DCEL^\sigma[P]_1(E, c) \leq c_j - DCEL^\sigma[P]_j(E, c)$ *for each* $j \neq 1$. *Let us suppose that there is* $\varphi \in \Phi[P]$ *such that* $\varphi_1(E, c) < DCEL^\sigma[P]_1(E, c)$. *By efficiency for some* $j \neq 1$ $\varphi_j(E, c) > DCEL^\sigma[P]_j(E, c)$. *Then,* $c_1 - \varphi_1(E, c) > c_j - \varphi_j(E, c)$, *contradicting* OP. *Hence,* $Pr_1(E, c) = DCEL^\sigma[P]_1(E, c)$. *Similarly,* $Pr_n(E, c) = DCEA^\sigma[P]_n(E, c)$.

Proof of Theorem 1.
Given $\sigma \in \Omega$ *and* P, *for each* $(E, c) \in \mathcal{D}$, *let* $S = \{r \in N | \ s_r(E^m, c^m) = CEL_r(E^m, c^m)$ *at some step* $m \in \mathbb{N}\}$ *and* $T = N \setminus S$. *By Lemma 5,* $Pr_1(E, c) = DCEL^\sigma[P]_1(E, c)$. *Furthermore, by Lemmas 2 and 4, for each agent* $r \in S$, *we have that* $\sum_{k=1}^{\infty} Pr_r(E^k, c^k) = DCEL^\sigma[P]_r(E, c)$. *Moreover, for each agent* $l \in T$, *by Lemma 3,* $\sum_{k=1}^{\infty} Pr_l(E^k, c^k) \leq DCEL^\sigma[P]_l(E, c)$.

Then, since $RD^\sigma[P](E, c)$ *exhausts the endowment* $R^\sigma[P](E, c) = DCEL^\sigma[P](E, c)$.

∎

Appendix 2 *Proof of Theorem 2*

The proof is based two lemmas. As mentioned, since F *and* F^d *should fulfil* P, *the set of PF-admissible rules is characterised by a set of self-dual axioms.*

Lemma 6 *For each* $P \in 2^{\mathcal{A}}$, $(E, c) \in \mathcal{D}$, *and* $m \in \mathbb{N}, m > 1$, $\sum_{i \in N} [Pr_i(E^m, c^m) + Pu_i(E^m, c^m)] = C^m$.

Proof.

Let $m \in \mathbb{N}, m > 1$. *Note that, for each* $P \in 2^{\mathcal{A}}$, $(E,c) \in \mathcal{D}$, $Pr(E,c) = c - Pu(L,c)$. *Then,* $\sum_{i \in N} \left[\frac{Pu_i(E^m,c^m)}{2} + \frac{Pr_i(E^m,c^m)}{2} \right] = E^m$.

Finally, $E^m = E^{m-1} - \sum_{i \in N} Pr_i(E^{m-1}, c^{m-1}) = \sum_{i \in N} \left[\frac{Pu_i(E^{m-1},c^{m-1})}{2} + \frac{Pr_i(E^{m-1},c^{m-1})}{2} \right]$ $- \sum_{i \in N} Pr_i(E^{m-1}, c^{m-1}) = \sum_{i \in N} \left[\frac{Pu_i(E^{m-1},c^{m-1}) - Pr_i(E^{m-1},c^{m-1})}{2} \right] = C^m/2$, *by the definition of the double recursive process.*

Lemma 7 *For each* $P \in 2^{\mathcal{A}}$, $(E,c) \in \mathcal{D}$ *and each* $i \in N$ *such that* $m \in \mathbb{N}, m > 1, c_i^m = Pu_i(E^m, c^m) + Pr_i(E^m, c^m)$.

Proof. *Let* $m \in \mathbb{N}$, $m > 1$. *Note that, for each* $P \in 2^{\mathcal{A}}$, $(E,c) \in \mathcal{D}$, $E^m = L^m = C^m/2$. *We know that* $L^m = C^m - E^m$. *By Lemma 6,* $E^m = C^m/2$. *Therefore,* $L^m = C^m - C^m/2 = C^m/2$. *For each* $i \in N$, $Pr_i(E^m, c^m) = Pr_i(L^m, c^m)$. *By duality,* $Pu_i(E^m, c^m) = c_i^m - Pr_i(L^m, c^m) = c_i^m - Pr_i(E^m, c^m)$, *then,* $c_i^m = Pu_i(E^m, c^m) + Pr_i(E^m, c^m)$. ■

Proof of Theorem 2.

For each $m \in \mathbb{N}$ *and each* $i \in N$, *we have two possibilities,*

$$DR^{\sigma}[P;F]_i\,(E,c) = \lfloor Pr_i(E,c) + \sum_{m=2}^{\infty} Pr_i(E^m, c^m) \rfloor + 1,$$

if i *is in the list of* E' *elements with highest priority ordering in* $Q\left(\frac{Pr(E,c)+Pu(E,c)}{2}; E, c\right)$, *where* $E' = E - \sum_{i \in N} \frac{Pr_i(E,c)+Pu_i(E,c)}{2} > 0$; *otherwise,*

$$DR^{\sigma}[P;F]_i\,(E,c) = \lfloor Pr_i(E,c) + \sum_{m=2}^{\infty} Pr_i(E^m, c^m) \rfloor.$$

By the definition of the double recursive discrete process,
$\sum_{m=2}^{\infty} c_i^m = \sum_{m=2}^{\infty} [Pu_i(E^{m-1}, c^{m-1}) - Pr_i(E^{m-1}, c^{m-1})] = Pu_i(E,c) + \sum_{m=2}^{\infty} Pu_i(E^m, c^m) - Pr_i(E,c) - \sum_{m=2}^{\infty} Pr_i(E^m, c^m)$.

By Lemma 7, $\sum_{m=2}^{\infty} c_i^m = \sum_{m=2}^{\infty} \left[Pu_i(E^m, c^m) + Pr_i(E^m, c^m) \right]$. *So,* $Pu_i(E,c) + \sum_{m=2}^{\infty} Pu_i(E^m, c^m) - Pr_i(E,c) - \sum_{m=2}^{\infty} Pr_i(E^m, c^m) = \sum_{m=2}^{\infty} [Pu_i(E^m, c^m) + Pr_i(E^m, c^m)]$. *Thus,* $\sum_{m=2}^{\infty} Pr_i(E^m, c^m) = \Big(Pu_i(E,c) - Pr_i(E,c) \Big)/2$.

Therefore, $DR^{\sigma}[P;F]_i\,(E,c) = \lfloor \frac{Pr_i(E,c)+Pu_i(E,c)}{2} \rfloor + 1 = \lfloor \frac{F[P]_i(E,c)+F[P]_i^d(E,c)}{2} \rfloor + 1$, *if* i *is in the list of* E' *elements with highest priority ordering in* $Q(\frac{Pr(E,c)+Pu(E,c)}{2}; E, c)$, *where* $E' = E - \sum_{i \in N} \frac{Pr_i(E,c)+Pu_i(E,c)}{2} > 0$; *otherwise,*

$$DR^{\sigma}[P;F]_i\,(E,c) = \lfloor \frac{Pr_i(E,c) + Pu_i(E,c)}{2} \rfloor .$$

Consequently, since $\lfloor \frac{F[P]_i(E,c)+F[P]_i^d(E,c)}{2} \rfloor = \lfloor \frac{Pr_i(E,c)+Pu_i(E,c)}{2} \rfloor$, $\lfloor \frac{F[P]_i(E,c)+F[P]_i^d(E,c)}{2} \rfloor = DR^{\sigma}[P;F]_i\,(E,c)$.

■

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