Economics Bulletin

Volume 35, Issue 4

On the Lorenz-maximal allocations in the imputation set

Francesc Llerena
Universitat Rovira i Virgili

Llúcia Mauri
Universitat Rovira i Virgili

Abstract

In this note we introduce the Lorenz stable set and provide an axiomatic characterization in terms of constrained egalitarianism and projection consistency. On the domain of all coalitional games, we find that this solution connects the weak constrained egalitarian solution (Dutta and Ray, 1989) with their strong counterpart (Dutta and Ray, 1991).

The first author acknowledges the support from research grants ECO2014-52340-P (Ministerio de Economia y Competitividad) and 2014SGR631 (Generalitat de Catalunya).
Citation: Francesc Llerena and Llúcia Mauri, (2015) "On the Lorenz-maximal allocations in the imputation set", *Economics Bulletin*, Volume 35, Issue 4, pages 2475-2481
Contact: Francesc Llerena - francesc.llerena@urv.cat, Llúcia Mauri - llucia.mauri@urv.cat.
Submitted: October 04, 2015. **Published:** November 21, 2015.

1. Introduction

One of the objectives of coalitional game theory is to define solutions (or rules) for allocating the joint profit arising from cooperation between a group of agents. In this framework, the first solution concept was introduced by von Neumann and Morgenstern (1944). There, a *stable set* is defined to be a subset of imputations satisfying *internal stability* and *external stability*, where the notion of stability is defined by means of a domination relation that uses the standard order in $\mathbb{R}$. Unfortunately, finding stable sets is a difficult task and neither existence nor uniqueness are guaranteed. In this note, we propose to combine the idea of internal and external stability with the Lorenz order. In this way, a set of imputations $\mathcal{V}$ is said to be *Lorenz stable* if it satisfies *internal Lorenz stability* (no element in $\mathcal{V}$ is Lorenz dominated by other element in $\mathcal{V}$) and *external Lorenz stability* (every element outside $\mathcal{V}$ is Lorenz dominated by some element in $\mathcal{V}$). Clearly, this definition leads to select the Lorenz-maximal allocations in the imputation set. Other set solution concepts, like the *core* (Gillies, 1959) or the *equal division core* (Selten, 1972), play a role in defining egalitarian solutions (see, for instance, Dutta and Ray, 1989, Dutta and Ray 1991, Hougaard et al., 2001 or Arin and Iñarra, 2001) but, as far as we know, the imputation set as a whole has not been considered to make egalitarian comparisons.

With this objective in mind, the paper is organized as follows. In Section 2 we introduce notation and terminology. Section 3 contains the main results. First we find that the Lorenz stable set is a singleton and admits an easy formula to be computed. We also provide an axiomatic characterization similar to the one given by Dutta (1990) to characterize the *weak constrained egalitarian solution* of Dutta and Ray (1989). Finally, in Section 4 we connects the Lorenz stable set with the *weak* and the *strong constrained egalitarian* solutions of Dutta and Ray (1989, 1991).

2. Notation and terminology

The set of natural numbers $\mathbb{N}$ denotes the universe of potential players. A **coalition** is a non-empty finite subset of $\mathbb{N}$ and let $\mathcal{N}$ denote the set of all non-empty coalitions of $\mathbb{N}$. A **transferable utility coalitional game (a game)** is a pair (N, v) where $N \in \mathcal{N}$ is the set of players and $v : 2^N \longrightarrow \mathbb{R}$ is the characteristic function that assigns to each coalition $S \subseteq N$ a real number $v(S)$, with the convention $v(\emptyset) = 0$. Given $S, T \in \mathcal{N}$, we use $S \subset T$ to indicate strict inclusion, that is $S \subseteq T$ but $S \neq T$. By $|S|$ we denote the cardinality of the coalition $S \in \mathcal{N}$. From now on we only consider games with at least two players. Thus, $\mathcal{N} := \{N \,|\, \emptyset \neq N \subseteq \mathbb{N},\ |N| \geq 2\}$. By Γ we denote the class of all games with $|N| \geq 2$.

Given $N \in \mathcal{N}$, let $\mathbb{R}^N$ stand for the space of real-valued vectors indexed by N, $x = (x_i)_{i \in N}$, and for all $S \subseteq N$, $x(S) = \sum_{i \in S} x_i$, with the convention $x(\emptyset) = 0$. For each $x \in \mathbb{R}^N$ and $T \subseteq N$, $x_{|T}$ denotes the restriction of x to T: $x_{|T} = (x_i)_{i \in T} \in \mathbb{R}^T$. Given two vectors $x, y \in \mathbb{R}^N$, $x \geq y$ if $x_i \geq y_i$, for all $i \in N$. We say that $x > y$ if $x \geq y$ and for some $j \in N$, $x_j > y_j$. Moreover, $x \gg y$ if $x_i > y_i$ for all $i \in N$. A set $\pi = \{P_1, \ldots, P_m\}$, where $P_i \subseteq N$ for all $i \in \{1, \ldots, m\}$, with $m \leq |N|$, is a **partition** of $N \in \mathcal{N}$ if the following conditions hold: (i) $P_i \neq \emptyset$ for all $i \in \{1, \ldots, m\}$, (ii) $\cup_{i=1}^m P_i = N$ and (iii) $P_i \cap P_j = \emptyset$, for all $i, j \in \{1, \ldots, m\}$, $i \neq j$. Given a game (N, v), a non-empty coalition T is an **equity coalition** if $\frac{v(S)}{|S|} \leq \frac{v(T)}{|T|}$ for all $\emptyset \neq S \subseteq T$.

The set of **feasible payoff vectors** of a game (N, v), with $N \in \mathcal{N}$, is defined by $X^*(N, v) := \{x \in \mathbb{R}^N \,|\, x(N) \leq v(N)\}$. A **solution** on a class of games $\Gamma' \subseteq \Gamma$, is a mapping σ which associates with every $N \in \mathcal{N}$ and every game $(N, v) \in \Gamma'$ a subset

$\sigma(N, v)$ of $X^*(N, v)$. Notice that σ is allowed to be empty. A solution σ is **single-valued** if for all $N \in \mathcal{N}$ and all $(N, v) \in \Gamma'$, $|\sigma(N, v)| = 1$. Then, we write $x = \sigma(N, v)$ instead of $\{x\} = \sigma(N, v)$. The **pre-imputation set** of (N, v) is defined by $X(N, v) := \{x \in \mathbb{R}^N \,|\, x(N) = v(N)\}$, and the set of **imputations** by $I(N, v) := \{x \in X(N, v) \,|\, x(i) \geq v(i),\ \text{for all } i \in N\}$. A game is **essential** if it has a non-empty imputation set. The **core** of (N, v) is the set of those imputations where each coalition gets at least its worth, that is $C(N, v) = \{x \in X(N, v) \mid x(S) \geq v(S) \text{ for all } S \subseteq N\}$. A game is **balanced** if it has a non-empty core. A game (N, v) is **superadditive** if $v(S) + v(T) \leq v(S \cup T)$ for all $S, T \subseteq N$ with $S \cap T = \emptyset$, and **convex** (Shapley, 1971) if, for every $S, T \subseteq N$, $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$.

Given $N \in \mathcal{N}$, for any $x \in \mathbb{R}^N$, denote by $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$ the vector obtained from x by rearranging its coordinates in a non-increasing order, that is, $\hat{x}_1 \geq \hat{x}_2 \geq \ldots \geq \hat{x}_n$. For any two vectors $y, x \in \mathbb{R}^N$ with $y(N) = x(N)$, we say that y **weakly Lorenz dominates** x, denoted by $y \succeq_{\mathcal{L}} x$, if $\sum_{j=1}^{k} \hat{y}_j \leq \sum_{j=1}^{k} \hat{x}_j$, for all $k \in \{1, \ldots, |N|\}$. We say that y **Lorenz dominates** x, denoted by $y \succ_{\mathcal{L}} x$, if at least one of the above inequalities is strict. Given a coalition $S \in \mathcal{N}$ and a set $A \subseteq \mathbb{R}^S$, EA denotes the set of allocations that are Lorenz undominated within A. That is, $EA := \{x \in A \mid \text{there is no } y \in A \text{ such that } y \succ_{\mathcal{L}} x\}$.

Given a game (N, v), the **weak Lorenz core** (Dutta and Ray, 1989) is defined in a recursive way as follows: the weak Lorenz core of a singleton coalition is $L(\{i\}, v) = \{v(i)\}$. Now suppose that the weak Lorenz core for all coalitions of cardinality k or less have been defined, where $1 < k < |N|$. The weak Lorenz core of a coalition $S \subset \mathbb{N}$ of size $(k+1)$ is defined by $L(S, v) = \{x \in \mathbb{R}^S \mid x(S) = v(S),\ \text{and there is no } T \subset S \text{ and } y \in EL(T, v) \text{ such that } y > x_{|T}\}$. The **weak constrained egalitarian solution**, denoted by EL, selects the vectors Lorenz-undominated within the weak Lorenz core. For $(N, v) \in \Gamma$, $|EL(N, v)| \leq 1$ (Dutta and Ray, 1989). The **strong Lorenz core** (Dutta and Ray, 1991) is defined in a similar way, but replacing $>$ by $\gg$. Dutta and Ray (1991) show that the strong Lorenz core, denoted by L^*, coincides with the **equal division core** when the coalition structure is N and there are no restrictions on coalition formation (see Selten, 1972 for details). That is, given an essential game (N, v), $L^*(N, v) = \{x \in I(N, v) \mid \text{for all } \emptyset \neq S \subset N,\ \text{there is } i \in S \text{ with } x_i \geq \frac{v(S)}{|S|}\}$. The **strong constrained egalitarian solution** chooses the vectors Lorenz-undominated within the strong Lorenz core. The **constrained egalitarian solution**, denoted by CE, is a single-valued solution defined for two person essential games as follows: let (N, v) be an essential game with $N = \{i, j\}$ and suppose, w.l.o.g., $v(i) \leq v(j)$, then

$$CE_j(N, v) = \max\left\{\frac{v(N)}{2}, v(j)\right\} \text{ and } CE_i(N, v) = v(N) - CE_j(N, v). \tag{1}$$

On the domain of convex games, the weak constrained egalitarian solution of Dutta and Ray (1989) selects the core allocation which Lorenz dominates every other point in the core. Moreover, it picks the payoff vector that is obtained by the following algorithm: Let (N, v) be a convex game and $EL(N, v) = \{z\}$.

- *Step 1:* Define $v_1 = v$. Then find the unique coalition $T_1 \subseteq N$ such that for all $T \subseteq N$, (i) $\frac{v_1(T_1)}{|T_1|} \geq \frac{v_1(T)}{|T|}$, and (ii) if $\frac{v_1(T_1)}{|T_1|} = \frac{v_1(T)}{|T|}$ and $T \neq T_1$, then $|T_1| > |T|$. Uniqueness of such a coalition is guaranteed by convexity of (N, v). For all $i \in T_1$, $z_i = \frac{v_1(T_1)}{|T_1|}$.
- *Step k:* Suppose that $T_1, \ldots, T_{k-1}$ have been defined.

- Let $N_k = N \setminus T_1 \cup \ldots \cup T_{k-1}$ and let (N_k, v_k) be the **marginal game** defined as follows: for all $S \subseteq N_k$,

$$v_k(S) := v(T_1 \cup \ldots \cup T_{k-1} \cup S) - v(T_1 \cup \ldots \cup T_{k-1}), \tag{2}$$

  It can be shown that (N_k, v_k) is convex. Then find the unique coalition $T_k \subseteq N_k$ such that for all $T \subseteq N_k$, (i) $\frac{v_k(T_k)}{|T_k|} \geq \frac{v_k(T)}{|T|}$, and (ii) if $\frac{v_k(T_k)}{|T_k|} = \frac{v_k(T)}{|T|}$ and $T \neq T_k$, then $|T_k| > |T|$. For all $i \in T_k$, $z_i = \frac{v_k(T_k)}{|T_k|} = \frac{v(T_1\cup\ldots\cup T_k) - v(T_1\cup\ldots\cup T_{k-1})}{|T_k|}$.

Given an essential game (N, v), for $X \subseteq I(N, v)$ we denote by $\mathcal{L}^v(X)$ the set of all imputations Lorenz dominated by some imputation of the set X. Formally, $\mathcal{L}^v(X) = \{y \in I(N, v) \,|\, \exists\, x \in X,\, x \succ_{\mathcal{L}} y\}$. A non-empty set of imputations $\mathcal{V} \subseteq I(N, v)$ is a *Lorenz stable set* for the game (N, v) if it satisfies the next two conditions:

1. $\mathcal{V}$ is *internally Lorenz stable*: no imputation in $\mathcal{V}$ Lorenz dominates another imputation in $\mathcal{V}$. Formally, $\mathcal{V} \cap \mathcal{L}^v(\mathcal{V}) = \emptyset$.
2. $\mathcal{V}$ is *externally Lorenz stable*: any imputation outside the set $\mathcal{V}$ is dominated by some imputation in $\mathcal{V}$. Formally, $\mathcal{V} \cup \mathcal{L}^v(\mathcal{V}) = I(N, v)$.

3. The Lorenz stable set

On the domain of essential games, we find that the Lorenz stable set is a singleton and admits a formula similar to that of the *constrained equal awards rule* for bankruptcy problems.

Definition 1. *Let (N, v) be an essential game. The vector $\mathbf{I}^v \in \mathbb{R}^N$ is defined as*

$$\mathbf{I}^v_i := \max\{v(i), \lambda\}, \tag{3}$$

for all $i \in N$, where λ is chosen so as to achieve efficiency.

Theorem 1. *Let (N, v) be an essential game. Then, there is a unique Lorenz stable set $\mathcal{V}$. Moreover, $\mathcal{V} = \{\mathbf{I}^v\}$.*

Proof of Theorem 1. Let (N, v) be an essential game with $N = \{1, \ldots, n\}$. Define the game (N, v^*) as follows: $v^*(S) = \sum_{i \in S} v(i)$ for all $S \subset N$, and $v^*(N) = v(N)$. Notice that (N, v^*) is convex and $C(N, v^*) = I(N, v)$. Since for convex games the egalitarian solution Lorenz dominates every other point in the core, we only need to check that $EL(N, v^*) = \{\mathbf{I}^v\}$. Assume, w.l.o.g, $v(1) \geq \ldots \geq v(n)$. If $v(1) \leq \frac{v^*(N)}{n}$, then $EL(N, v^*) = \left\{\mathbf{I}^v = \left(\frac{v^*(N)}{n}, \ldots, \frac{v^*(N)}{n}\right)\right\}$. Otherwise, take $k \in \{1, \ldots, n-1\}$, $n \geq 2$, and define the vector $y^k \in \mathbb{R}^N$ as follows,

$$y^k := \big(v(1), \ldots, v(k), \frac{v(N) - (v(1) + \ldots + v(k))}{n-k}, \ldots, \frac{v(N) - (v(1) + \ldots + v(k))}{n-k}\big).$$

Observe that $\mathbf{I}^v = y^{k^*}$, where $k^* = \min\{k \in \{1, \ldots, n-1\} \,|\, y^k_i \geq v(i) \text{ for all } i \in N\}$. Let $\mathcal{P} = \{S_1, \ldots, S_m\}$ be the partition of N generated by the Dutta and Ray (1989) algorithm to compute $EL(N, v^*)$. Denote $EL(N, v^*) = \{z\}$. Notice that $m \geq 2$ because $v(1) > \frac{v^*(N)}{n}$. It can be easily checked that $z_i = v(i)$ for all $i \in S_h$ and all $h \in \{1, \ldots, m-1\}$, and $z_i = \frac{v(N) - \sum_{i \in N \setminus S_m} v(i)}{|S_m|}$ for all $i \in S_m$. Hence, $z = y^k$ where $k = |S_1 \cup \ldots \cup S_{m-1}|$. Suppose $k > k^*$. By the minimality of k^*, we have $z_i \leq y^{k^*}_i$ for all $i \in \{1, \ldots, k^*, \ldots, k\}$. Moreover, for all $i > k$, since $i \in S_m$ and $k \in S_{m-1}$, we have $z_i < z_k = v(k) \leq y^{k^*}_k = y^{k^*}_i$. Then, $z(N) < y^{k^*}(N) = v(N)$, a contradiction. Hence, $k = k^*$ and $EL(N, v^*) = \{\mathbf{I}^v\}$. □

From Theorem 1 and the characterization of Lorenz domination given by Hardy et al. (1934),[1] it follows that the Lorenz stable solution selects the allocation in the imputation set that minimize the Euclidean distance to the equal division payoff vector. Formally, for all essential game (N, v),

$$\mathbf{I}^v = \arg \min_{x \in I(N,v)} \sum_{i \in N} \left(x_i - \frac{v(N)}{|N|} \right)^2 . \tag{4}$$

Next we introduce the properties that we will use to characterize axiomatically the Lorenz stable set. All of them have been used upon several times in the literature.

Let us denote by Γ_{es} the set of all essential games. Let σ be a solution on Γ_{es}, we say that σ satisfies:

- **Efficiency (EFF)** if for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma_{es}$ and all $x \in \sigma(N, v)$, it holds $x(N) = v(N)$.
- **Constrained egalitarianism (CE)** if for all $N \in \mathcal{N}$ with $|N| = 2$, and all $(N, v) \in \Gamma_{es}$, it holds $\sigma(N, v) = CE(N, v)$.
- **Projection consistency (PCONS)** if for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma_{es}$, all $x \in \sigma(N, v)$ and all $\emptyset \neq T \subset N$, it holds $(T, r_x(v)) \in \Gamma_{es}$ and $x_{|T} \in \sigma(T, r_x(v))$, where $(T, r_x(v))$ is the *projected reduced game* of (N, v) *relative to* x *and* T defined as follows:

$$r_x(v)(S) := \begin{cases} v(S) & \text{if } S \subset T, \\ v(N) - x(N \setminus T) & \text{if } S = T. \end{cases} \tag{5}$$

On the domain of convex games, Dutta (1990) characterizes the weak constrained egalitarian solution (Dutta and Ray, 1989) by means of constrained egalitarianism and max-consistency, that is, consistency with respect to the Davis and Maschler (1965) reduced game. On the domain of essential games, by replacing max-consistency by projection consistency we characterize the Lorenz stable set.[2]

Theorem 2. *On the domain of essential games, the only single-valued solution satisfying* **CE** *and* **PCONS** *is the Lorenz stable set.*

Proof of Theorem 2. **CE** is obvious. Next we prove **PCONS**. Let $N \in \mathcal{N}$, $(N, v) \in \Gamma_{es}$, $x = \mathbf{I}^v$ and $(T, r_x(v))$ be the projected reduced game associated to $\emptyset \neq S \subset N$ and x. Since $x_{|T} \in I(T, r_x(v))$, we have $(T, r_x(v)) \in \Gamma_{es}$. Let $y = \mathbf{I}^{r_x(v)}$ be the Lorenz stable set of $(T, r_x(v))$ and suppose $y \neq x_{|T}$. Then, $y \succ_{\mathcal{L}} x_{|T}$. Now consider the vector $z = (x_{|N \setminus T}, y) \in \mathbb{R}^N$. Since $z \in I(N, v)$, $x \succ_{\mathcal{L}} z$, which implies $x_{|T} \succ_{\mathcal{L}} y$, a contradiction.[3] Hence, $x_{|T} = y$. To prove uniqueness, let σ be a single-valued solution on Γ_{es} satisfying **CE** and **PCONS**. For $|N| = 2$, uniqueness follows from **CE**. Let $(N, v) \in \Gamma_{es}$ with $N = \{1, 2, \ldots, n\}$, $n \geq 3$,

[1] If x and y are two vectors in $\mathbb{R}^n$ with $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, the following statements are equivalent: (a) x Lorenz dominates y; (b) for any strictly concave function $U : \mathbb{R} \longrightarrow \mathbb{R}$, we have $\sum_{i=1}^n U(x_i) > \sum_{i=1}^n U(y_i)$.

[2] Projection consistency has been used to characterize, among others, the equal division core (Bhattacharya, 2004) or the undominated core (Llerena and Rafels, 2007).

[3] Let N be a finite set of players, and let $S \subseteq N$, $S \neq \emptyset$. If x_S, $y_S \in \mathbb{R}^S$, $x_S(S) = y_S(S)$, and $z_{N \setminus S} \in \mathbb{R}^{N \setminus S}$, then x_S Lorenz dominates y_S if and only if $(x_S, z_{N \setminus S})$ Lorenz dominates $(y_S, z_{N \setminus S})$. This remark is stated in Hougaard et al. (2001) page 153, and it is based on Theorem 108 of Hardy et al. (1934).

and $x=\sigma(N,v)$. First observe that **CE** and **PCONS** imply **EFF**, that is, $x(N)=v(N)$. Let $T=\{i,j\}\subset N$. By **CE** and **PCONS**, $x_{|T}=\sigma(T,r_x(v))=CE(T,r_x(v))$. Thus, $x\in I(N,v)$. If $x_1=\ldots=x_n$, then $x=\mathbf{I}^v$. Otherwise, suppose, w.l.o.g., $x_1>\ldots>x_{k+1}=\ldots=x_n$, for some $k\in\{1,\ldots,n-1\}$. For $i\in\{1,\ldots,k\}$, let $T=\{i,i+1\}$. By **PCONS**, $x_{|T}=CE(T,r_x(v))$. Since $x_i>x_{i+1}$, $x_i=v(i)$ for all $i\in\{1,\ldots,k\}$. Now, by **EFF** we obtain $x_i=\frac{v(N)-(v(1)+\ldots+v(k))}{n-k}$ for all $i\in\{k+1,\ldots,n\}$. Thus, for all $i\in N$, $x_i=\max\{v(i),\lambda\}$ being $\lambda=\frac{v(N)-(v(1)+\ldots+v(k))}{n-k}$, and $x=\mathbf{I^v}$. □

The axioms in Theorem 2 are independent. For instance, the solution σ_1 defined, for all $N\in\mathcal{N}$ and all $(N,v)\in\Gamma_{es}$, as $\sigma_1(N,v)=\left(\frac{v(N)}{|N|},\ldots,\frac{v(N)}{|N|}\right)$, satisfies **PCONS** but not **CE**. The solution σ_2 defined, for all $N\in\mathcal{N}$ and all $(N,v)\in\Gamma_{es}$, as $\sigma_2(N,v)=CE(N,v)$ if $|N|=2$, and $\sigma_2(N,v)=(v(i))_{i\in N}$ otherwise, satisfies **CE** but not **PCONS**.

4. Connecting the egalitarian solutions of Dutta and Ray (1989, 1991)

Dutta and Ray (1991) characterize the class of superadditive games in which the weak constrained egalitarian allocation (Dutta and Ray, 1989) and their strong counterpart (Dutta and Ray, 1991) coincide. Here we show that, on the domain of all games, the unique weak constrained egalitarian allocation happens to be a strong if and only if the two set of allocations are singleton containing the Lorenz stable allocation. Consequently, for superadditive games we find an easy way to check when coincidence occurs.

Theorem 3. *Let (N,v) be an game. Then, the following statements are equivalent:*

(i) $EL(N,v)\cap EL^*(N,v)\neq\emptyset$.

(ii) $EL(N,v)=\{\mathbf{I}^v\}$.

(iii) $EL(N,v)=EL^*(N,v)\neq\emptyset$.

Proof of Theorem 3. (i) $\Rightarrow$ (ii): Let $EL(N,v)\cap EL^*(N,v)=\{y\}$ and let us assume, w.l.o.g., that $y_1\geq y_2\geq\ldots\geq y_n$. If $y_1=y_n$, then $y=\left(\frac{v(N)}{|N|},\ldots,\frac{v(N)}{|N|}\right)$ and so $y=\mathbf{I}^v$. If $y_1>y_n$, then $T=\{i\in N\,|\,y_i>y_n\}\neq\emptyset$. Let $j^*\in T$, by Lemma 2 of Dutta and Ray (1991)[4] there exists an equity coalition R containing j^* and such that $\frac{v(R)}{|R|}=y_{j^*}$ and $R\subset\{i\in N\,|\,y_i<y_{j^*}\}\cup\{j^*\}$. If $|R|=1$, then $y_{j^*}=v(j^*)$. Otherwise, if $|R|\geq 2$, then $EL(R,v)=\left\{\left(\frac{v(R)}{|R|},\ldots,\frac{v(R)}{|R|}\right)\right\}$. Since $y\in EL(N,v)$ there exists $i^*\in R$ such that $y_{i^*}>\frac{v(R)}{|R|}=y_{j^*}$, getting a contradiction. Then $R=\{j^*\}$. Thus, $y_i=v(i)$ for all $i\in T$ and, by efficiency, $y_i=\frac{v(N)-\sum_{j\in T}v(j)}{|N|-|T|}$, for all $i\in N\setminus T$. We know that

$$\mathbf{I}^v=\left(v(1),\ldots,v(k),\frac{v(N)-\sum_{i=1}^{k}v(i)}{n-k},\ldots,\frac{v(N)-\sum_{i=1}^{k}v(i)}{n-k}\right),$$

where $k=\min\left\{j\in N\,|\,\frac{v(N)-\sum_{i=1}^{j}v(i)}{n-j}\geq v(j+1)\right\}$. Since $y\in I(N,v)$, $|T|=t\geq k$. Suppose $t>k$. For all $i\in\{1,\ldots,k\}$, $\mathbf{I}_i^v=y_i=v(i)$, for all $i\in\{k+1,\ldots,t\}$, $\mathbf{I}_i^v\geq$

[4] Lemma 2 in Dutta and Ray (1991) states the following: *For some $S\subseteq N$, let $y\in EL^*(S,v)$. For any $i\in S$, if $y_i>\min_{j\in S}y_j$, then there exists an equity coalition T containing i and satisfying: (i) $\frac{v(T)}{|T|}=y_i$ and (ii) $T\subset\{k\in S\,|\,y_k<y_i\}\cup\{i\}$.*

$v(i) = y_i$, and for all $i \in \{t+1, \ldots, n\}$, $\mathbf{I}^v_i = \mathbf{I}^v_t \geq v(t) = y_t > y_i$. But then, $v(N) = \mathbf{I}^v(N) > y(N)$ in contradiction with $y(N) = v(N)$. Hence, $k = t$ and $y = \mathbf{I}^v$.

The implication (ii) $\Rightarrow$ (iii) follows from $L(N, v) \subseteq L^*(N, v) \subseteq I(N, v)$ and the fact that $\mathbf{I}^v$ Lorenz dominates every other point in the imputation set. Obviously (iii) $\Rightarrow$ (i). □

As a consequence of Theorem 3 we obtain the following corollary for superadditive games.

Corollary 1. *Let (N, v) be a superadditive game. Then, the following statements are equivalent:*

(i) $EL(N, v) = EL^*(N, v)$.

(ii) $\mathbf{I}^v \in C(N, v)$.

Proof of Corollary 1. Notice first that for superadditive games, $EL^*(N, v) \neq \emptyset$. From Theorem 3, $EL(N, v) = EL^*(N, v) \neq \emptyset$ implies $EL(N, v) = EL^*(N, v) = \{\mathbf{I}^v\}$. On this domain, both solution coincide when the unique strong constrained egalitarian allocation belongs to the core (Dutta and Ray, 1991), thus $\mathbf{I}^v \in C(N, v)$. Conversely, since $C(N, v) \subseteq L(N, v) \subseteq L^*(N, v)$ and $\mathbf{I}^v$ Lorenz dominates every other point in the imputation set, we have $EL(N, v) = EL^*(N, v) = \{\mathbf{I}^v\}$. □

References

Arin, J.; Iñarra, E. (2008) "Egalitarian solutions in the core" *Int. J. Game Theory*, *30*, 187-193.

Bhattacharya, A. (2004) "On the equal division core" *Soc. Choice Welfare*, *22*, 391-399.

Davis, M.; Maschler M. (1965) "The kernel of a cooperative game" *Nav. Res. Logist. Q.*, *12*, 223-259.

Dutta, B. (1990) "The egalitarian solution and reduced game properties in convex games" *Int. J. Game Theory*, *19*, 153-169.

Dutta, B.; Ray, D. (1989) "A concept of egalitarianism under participation constraints" *Econometrica*, *57*, 615-635.

Dutta, B.; Ray, D. (1991) "Constrained egalitarian allocations" *Game. Econ. Behav.*, *3*, 403-422.

Gillies, D. (1959) "Solutions to general non-zero-sum games" In *Ann. Math. Stud.*, *40*, Tucker, A.W.; Luce, R.D. (editors),*Contributions to the Theory of Games* vol. IV, 47-58, Princeton University Press, Princeton.

Hardy, G.H.; Littlewood, J.E.; Pólya, G. (1934) "Inequalities" In *Cambridge University Press*, London.

Hougaard, J.L.; Peleg, B.; Thorlund-Petersen, L. (2001) "On the set of Lorenz-maximal imputations in the core of a balanced game" *Int. J. Game Theory*, *30*, 147-165.

Llerena, F.; Rafels C. (2007) "Convex decomposition of games and axiomatizations of the core and the D-core" *Int. J. Game Theory*, *35*, 603-615.

Selten, R. (1972) "Equal share analysis of characteristic function experiments" In *Contributions to Experimental Economics III 3*, Sauermann, H. (editors); Mohr, Tubingen; pp.130-165.

Shapley, L.S. (1971) "Cores of convex games" *Int. J. Game Theory*, *1*, 11-16.

Von Neumann, J.; Morgenstern, O. (1944) "Theory of Games and Economic Behavior" In *Princeton University Press*, Princeton.