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A Note on Borda Method

Surekha K Rao
Indiana University Northwest

Bhaskara Rao Kopparty
Indiana University Northwest

Abstract

The Borda method is a form of preference voting where the rankings by the voters are converted into scores using weights for the ranks, and the society's preference order on the alternatives is determined by the order on the scores obtained by each alternative. In this paper we show that once the individuals' preference orders are given there is at least one preference order that cannot be the societal preference order whatever is the choice of weights for the ranks.

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Contact: Surekha K Rao - skrao@iun.edu, Bhaskara Rao Kopparty - bkoppart@iun.edu.
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1 Introduction

The Borda method is a form of preference voting where the rankings by the voters are converted into scores using weights for the ranks, and the society's preference order on the alternatives is determined by the order on the scores obtained by each alternative. In order to decide the societal preference order one can take any system of weights $\omega_1 \geq \omega_2 \geq \cdots \geq \omega_n$ as the weights for $rank\ 1, rank\ 2, rank\ 3, \cdots rank\ n$. Different systems of weights can produce different societal preference orders for a given collection of preference orders by the voters. One can also use the system of weights to decide the winner(s) of an election - the alternative(s) that gets the highest score is declared the winner, or the loser(s) of the election - the alternative that gets the lowest score is the loser.

In Nauru the weight system $\omega_i = n - i$ for $i = 1, 2, \cdots, n$ is used in determining the societal preference order of the candidates. In Kirabati the weight system $\omega_i = 1/i$ for $i = 1, 2, \cdots, n$ is used in their elections. Though these weights look like naturally occuring numbers, really any system of weights (as long as the weights are decreasing for decreasing ranks) can be used for determining the scores of each of the alternatives. We shall call this the *Borda Method* of determining the societal preference order on the alternatives or for determining the winner or the loser.

Let us look at an example. Consider the preference order profile expressed by 10 voters on alternatives A_1, A_2, and A_3 as follows. 2 voters rank A_1 as $rank\ 1, A_2$ as $rank\ 2, A_3$ as $rank\ 3$; 3 voters rank A_1 as $rank\ 1, A_3$ as $rank\ 2, A_2$ as $rank\ 3$; 4 voters rank A_2 as $rank\ 1, A_3$ as $rank\ 2, A_1$ as $rank\ 3$, and 1 voter ranks A_3 as $rank\ 1, A_2$ as $rank\ 2$, and A_1 as $rank\ 3$.

If we consider the columns to represent the alternatives and rows to represent the ranks, the frequencies can be written in the matrix notation for this preference profile as

$$M = \begin{bmatrix} 5 & 4 & 1 \\ 0 & 3 & 7 \\ 5 & 3 & 2 \end{bmatrix}.$$

For example, 7 in the second row, third column of this matrix represents the number of voters that ranked A_3 as rank 2.

If we give weights $2, 1, 0$ for ranks $1, 2,$ and 3 resectively, as in Nauru, the scores obtained by A_1, A_2, and A_3 are $10, 11$ and 9 respectively. We may convert these scores to societal rankings by saying that the society ranks A_2 as $rank\ 1$, A_1 as $rank\ 2$, A_3 as $rank\ 3$ and the society ranks A_2 strictly higher thanA_1 and A_1 strictly higher than A_3. If we are selecting the winner using Borda method with these weights, A_2 will be the winner(no ties for the winner). If we are selecting the loser using Borda method with these weights, A_3 will be the loser(no ties for the loser). If we are deciding the societal preference order, it would be $A_2 \succ A_1 \succ A_3$.

Let us now use the weights $1, 1/2, 1/3$ as in Kirabati. The scores obtained by A_1, A_2, and A_3 would be $40/6, 39/6$ and $31/6$. If we are selecting the winner(loser) using Borda method with these weights, $A_1(A_3)$ will be the sole winner(sole loser). If we are deciding the societal preference order, it would be $A_1 \succ A_2 \succ A_3$.

Now, for this example, is it possible to engineer the weights so that $A_3 \succ A_1 \succ A_2$ is the societal preference order? Is to possible to engineer the weights so that A_2 is the loser? In this note we shall answer some questions of this type.

Let there be m voters in a society and assume that the voters give their preference orders on n alternatives $A_1, A_2, \cdots A_n$. The preference order of a voter is expressed by a ranking of the alternatives. Let us write $R_1, R_2, \cdots R_n$ for the ranks Rank 1, Rank 2, $\cdots$, Rank n. Every voter expresses her opinion by giving a preference order on the alternatives. For example, when there are 4 alternatives, A_1, A_2, A_3, A_4, the preference order of a voter could be A_2 is ranked R_1, A_4 is ranked R_2, A_1 is ranked R_3 and A_3 is ranked R_4.

We shall call the preference orders expressed by all the voters a preference order profile. For convenience, for the society we shall ask questions about a strict preference order being the societal preference order.

We shall address the question as to whether there are preference order profiles so that by manipulating the weights one can obtain any given strict preference order as the societal preference order using Borda method.

We shall show that the answer is in the negative in the sense that, whatever preference order profile we take, there is at least one strict preference order which is not the societal preference order for any weight system.

In the third section we shall return to the example and identify all possible winners, all possible losers and all possible strict societal preference orders.

Saari ([2], [3]) used the pairwise preference matrix to do an extensive study of the Borda method. Here we shall use another matrix which we shall call the RAF matrix. Let $M = (m_{ij})$ be the $m \times m$ matrix, where m_{ij} is the number of voters that assigned the i^{th} rank to A_j . We shall call this matrix the ranks-alternatives-frequencies matrix (RAF matrix) for the preference order profile. The matrix M summarizes all the preference orders expressed by all the voters. M is a nonnegative integer matrix and each row sum of M is m and each column sum of M is m. This is because every alternative is assigned some ranking by every voter and each rank is assigned to some alternative by every voter. It is possible that two preference order profiles may result in the same M. For the purpose of Borda method, M has all the information from the preference order profile. In fact any such M comes from a preference order profile. This can be shown by induction on m. There may be more than one preference order profile giving rise to the same RAF matrix.

Let $\underline{\omega}^T = (\omega_1, \omega_2, \cdots, \omega_n)$ with $\omega_1 \geq \omega_2 \geq \cdots \geq \omega_n \geq 0$ be the weights for ranks $R_1, R_2, \cdots, R_n$ respectively. We shall call this a weight system. We use these weights to find the scores $t_j's$ for each of the alternatives $A_1, A_2, \cdots, A_n$ by calculating the score for the alternative A_j as $t_j = \sum_{i=1}^{n} \omega_i m_{ij}$. Now, the societal ranking of the alternatives is determined by the order of $t_j's$. That is, we say that A_j is preferred to A_k (and write $A_j \succeq A_k$) by the society if $t_j \geq t_k$ and A_j is strictly preferred to A_k (and write $A_j \succ A_k$)

by the society if $t_j > t_k$. If all ω_j's are equal, all t_j's are equal and there is a tie between all the alternatives. For simiplicity let us assume that not all ω_j's are equal to each other.
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If we let $\underline{t}^T = (t_1, t_2, \cdots t_n)$, then $\underline{t}^T = \underline{\omega}^T M$.

The scores and as a consequence, the societal preference order, depend on the weight system that is chosen. Different weight systems would give different societal preference orders for the same voter preference profile. Once the voters of the society express their preference order profile, for every given strict preference order on the alternatives is it possible to choose a weight system so that the societal preference order is the prescribed preference order on the alternatives? That is, are there voter preference profiles so that each possible strict preference order can be achieved as the societal preference order by some weight system?

In the next section we shall prove our main result.

2 Main Result

In this section we shall show that once the voters' preference order profile is known there are some strict orderings of the alternatives that cannot be achieved by any weight system.

Theorem: Given any preference order profile, there exist some preference orders that are not the Borda method societal preference orders for any weight system.

Proof: Let M be the RAF matrix given by the preference order profile.

We introduce some notation. If $\underline{\omega}^T = (\omega_1, \omega_2, \cdots, \omega_n)$ where $\omega_1 \geq \omega_2 \geq \cdots \geq \omega_n \geq 0$ then $\underline{\omega}^T = \underline{\gamma}^T L$ where $\underline{\gamma}^T = (\gamma_1, \gamma_2, \ \cdots \gamma_n)$, all the $\gamma_i' s$ for $i = 1$ to $n-1$ are ≥ 0 and L is the lower triangular matrix with $0's$ above the main diagonal and $1's$ elsewhere.

Then, $\underline{t}^T = \underline{\omega}^T M = \underline{\gamma}^T LM$. Let us write $P = LM$. For the matrix P the first row of P is the first row of M, the second row of P is the sum of the first two rows of M, the third row of P is the sum of the first three rows of M, etc,. The last row of P is $(n, n, \cdots, n)$.

If we write $\underline{p_i}^T$ for the $i'th$ row of P, then, $\underline{t}^T = \gamma_1 \underline{p_1}^T + \gamma_2 \underline{p_2}^T + \cdots + \gamma_n \underline{p_n}^T$. If we write p_{ij} for the $j'th$ element of $\underline{p_i}^T$, $t_j = \sum_{i=1}^{n} \gamma_i p_{ij}$. For the society, A_j is preferred to A_k if and only if $t_j \geq t_k$. But $t_j = \sum_{i=1}^{n} \gamma_i p_{ij} \geq t_k = \sum_{i=1}^{n} \gamma_i p_{ik}$ happens if and only if $\sum_{i=1}^{n-1} \gamma_i p_{ij} \geq \sum_{i=1}^{n-1} \gamma_i p_{ik}$. Hence the societal preference order is determined by the $(m-1) \times m$ matrix Q consisting of the first $m-1$ rows of P. Indeed, if we call $(\gamma_1, \gamma_2, \cdots \gamma_{n-1})$ as $\underline{\eta}^T$, and $\underline{s}^T = (s_1, s_2, \cdots, s_n) = \underline{\eta}^T Q$ then $s_j \geq s_k$ if and only if $t_j \geq t_k$ and $s_j > s_k$ if and only if $t_j > t_k$.

Now, find a nonzero vector $\underline{c}^T = (c_1, c_2, \cdots c_n)$ so that the inner products $(\underline{p_j}, \underline{c})$ for $j = 1$ to $n-1$ are all equal to zero. Thus, $Q\underline{c} = \underline{0}$. We assume without loss of generality that $c_i > 0$ for at least one i. Since the vectors $\underline{p_1}, \underline{p_2}, \cdots \underline{p_{n-1}}$ are all nonnegative nonzero vectors, some of the $c_i' s$ are ≤ 0 also. $D = \{i : c_i > 0\}$, and $E = \{i : c_i \leq 0\}$, then D and E are nonempty. Let us write $d_i = -c_i$ if $c_i \leq 0$.

We shall now identify some preference orders that are not the societal preference orders for any weight system.

Consider the sums $S = \sum_{\{i \in D\}} c_i$ and $T = \sum_{\{i \in E\}} d_i$. $S > 0$ and $T \geq 0$.

Since S and T are not defined symmetrically we we need to look at two cases: $S \leq T$ and $T < S$.

Suppose that $S \leq T$. Here is a description of some of the rankings of alternatives that cannot be realized as societal preference orders for any weight system. Let R be any ranking of the alternatives in which A_i for every i in E is ranked strictly above A_j for every j in D. Since D and E are nonempty, there are preference orders satisfying this condition.

Suppose that a ranking as above is the societal preference order because of a weight system $\underline{\omega}$. Then by the argument above there is a nonnegative vector $\underline{\eta}^T = (\gamma_1, \gamma_2, \cdots \gamma_{n-1})$ such that $\gamma_i's$ for $i = 1$ to $n-1$ are ≥ 0 with at least one $\gamma_i > 0$ and the societal ranking is given by the order on $\underline{\eta}^T Q = \underline{s}^T = (s_1, s_2, \cdots, s_n)$, and the ordering of $s_i's$ agrees with the ranking R.

Since $Q\underline{c} = \underline{0}$, we have $\underline{s}^T \underline{c} = 0$. That is $\sum s_i c_i = 0$. Hence $\sum_{\{i \in D\}} s_i c_i = \sum_{\{i \in E\}} s_i d_i$.

Let i_0 be one of the indices from D such that $s_{i_0} = max\{s_i : i \in D\}$. Let i_1 be one of the indices from E such that $s_{i_1} = min\{s_i : i \in E\}$. Then, since we want A_{i_1} to be strictly preferred to A_{i_0}, s_{i_1} should be $> s_{i_0}$.

Now, $\sum_{\{i \in D\}} s_i c_i \leq (\sum_{\{i \in D\}} c_i) s_{i_0} = S \cdot s_{i_0}$ and $T \cdot s_{i_1} = (\sum_{\{i \in E\}} d_i) s_{i_1} \leq \sum_{\{i \in E\}} s_i d_i$. Since $0 \leq s_{i_0} < s_{i_1}$ and $0 < S \leq T$, $S \cdot s_{i_0} < T \cdot s_{i_1}$. Hence $\sum_{\{i \in D\}} s_i c_i < \sum_{\{i \in E\}} s_i d_i$. This is a contradiction to $\underline{s}^T \underline{c} = 0$.

Thus there is no weight system resulting in R as the societal ranking.

We shall now look at the case of $T < S$. Here is a description of some of the rankings of alternatives that cannot be realized as societal preference orders for any weight system. Let R be any ranking of the alternatives in which A_i for every i in D is ranked strictly above A_j for every j in E. Since D and E are nonempty, there are preference orders satisfying this condition.

Suppose that a ranking as above is the societal preference order because of a weight system $\underline{\omega}$. Then by the argument above there is a nonnegative vector $\underline{\eta}^T = (\gamma_1, \gamma_2, \cdots \gamma_{n-1})$ such that $\gamma_i's$ for $i = 1$ to $n-1$ are ≥ 0 with at least one $\gamma_i > 0$ and the societal ranking is given by the order on $\underline{\eta}^T Q = \underline{s}^T = (s_1, s_2, \cdots, s_n)$, and the ordering of $s_i's$ agrees with the ranking R.

Since $Q\underline{c} = \underline{0}$, we have $\underline{s}^T \underline{c} = 0$. That is $\sum s_i c_i = 0$. Hence $\sum_{\{i \in E\}} s_i c_i = \sum_{\{i \in D\}} s_i d_i$.

Let i_0 be one of the indices from E such that $s_{i_0} = max\{s_i : i \in E\}$. Let i_1 be one of the indices from D such that $s_{i_1} = min\{s_i : i \in D\}$. Then, since we want A_{i_1} to be strictly preferred to A_{i_0}, s_{i_1} should be $> s_{i_0}$.

Now, $\sum_{\{i \in E\}} s_i c_i \leq (\sum_{\{i \in E\}} c_i) s_{i_0} = T \cdot s_{i_0}$ and $S \cdot s_{i_1} = (\sum_{\{i \in D\}} d_i) s_{i_1} \leq \sum_{\{i \in D\}} s_i d_i$. Since $0 \leq s_{i_0} < s_{i_1}$ and $0 \leq T < S$, $T \cdot s_{i_0} < S \cdot s_{i_1}$. Hence $\sum_{\{i \in E\}} s_i d_i < \sum_{\{i \in D\}} s_i c_i$. This is a contradiction to $\underline{s}^T \underline{c} = 0$.

3 Examples

Example 1: Returning to the example at the beginning, we have

$$M = \begin{bmatrix} 5 & 4 & 1 \\ 0 & 3 & 7 \\ 5 & 3 & 2 \end{bmatrix}.$$

be the RAF matrix for some preference order profile for three alternatives A_1, A_2, A_3. Then

$$P = LM = \begin{bmatrix} 5 & 4 & 1 \\ 5 & 7 & 8 \\ 10 & 10 & 10 \end{bmatrix}$$

and

$$Q = \begin{bmatrix} 5 & 4 & 1 \\ 5 & 7 & 8 \end{bmatrix}$$

A vector $\underline{c}^T$ so that $Q\underline{c} = 0$ is given by $c = (-5/3, 7/3, -1)$. Then, $D = \{2\}, E = \{1, 3\}$, $S = 7/3$ and $T = 8/3$. Here $S \leq T$. Hence the strict preference orders $A_3 \succ A_1 \succ A_2$ and $A_1 \succ A_3 \succ A_2$ cannot be achieved for any ω of decreasing nonnegative weights. On the other hand, it so happens that all other preference orderings are achieved for some weight systems.

$(\omega_1, \omega_2, \omega_3)$	(γ_1, γ_2)	*scores for* A_1, A_2, A_3	*societal ranking*
$(2, 1, 0)$	$(1, 1)$	$(10, 11, 9)$	$A_2 \succ A_1 \succ A_3$
$(4, 1, 0)$	$(3, 1)$	$(20, 19, 11)$	$A_1 \succ A_2 \succ A_3$
$(3, 2, 0)$	$(1, 2)$	$(15, 18, 17)$	$A_2 \succ A_3 \succ A_1$
$(5, 4, 0)$	$(1, 4)$	$(25, 32, 33)$	$A_3 \succ A_2 \succ A_1$

Example 2: We shall look at another example, rather trivial, and see how our result produces preference orders that are not the preference orders for any weight systems.

Consider the preference order profile expressed by three voters on alternatives A_1, A_2, and A_3 as follows. All three voters rank A_1 as *rank* 3, A_2 as *rank* 2, A_3 as *rank* 1. Clearly all other preference orders are not the preference orders for any weight system. Let us see the conclusions we can obtain from our results. The RAF matrix M is give by

$$M = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 3 & 0 \\ 3 & 0 & 0 \end{bmatrix}.$$

Then

$$P = LM = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$$

and

$$Q = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 3 & 3 \end{bmatrix}$$

A vector $\underline{c}^T$ so that $Q\underline{c} = 0$ and some $c_i > 0$ is given by $c = (1, 0, 0)$. Then, $D = \{1\}, E = \{2, 3\}$, $S = 1$ and $T = 0$. Here $T < S$. Hence, according to the proof of the theorem the strict preference orders $A_1 \succ A_2 \succ A_3$ and $A_1 \succ A_3 \succ A_2$ cannot be achieved for any ω of decreasing nonnegative weights.

Example 3: We shall look a third example suggested by the referee. Consider the preference order profile expressed by ten voters on alternatives A_1, A_2, and A_3 as follows. Eight voters rank A_1 as *rank* $1, A_2$ as *rank* $2, A_3$ as *rank* 3. Two voters rank A_1 as *rank* $2, A_2$ as *rank* $1, A_3$ as *rank* 3.

The RAF matrix M is give by

$$M = \begin{bmatrix} 8 & 2 & 0 \\ 2 & 8 & 0 \\ 0 & 0 & 10 \end{bmatrix}.$$

Then

$$P = LM = \begin{bmatrix} 8 & 2 & 0 \\ 10 & 10 & 0 \\ 10 & 10 & 10 \end{bmatrix}$$

and

$$Q = \begin{bmatrix} 8 & 2 & 0 \\ 2 & 8 & 0 \end{bmatrix}$$

A vector $\underline{c}^T$ so that $Q\underline{c} = 0$ and some $c_i > 0$ is given by $c = (0, 0, 1)$. Then, $D = \{3\}, E = \{1, 2\}$, $S = 1$ and $T = 0$. Here $T < S$. Hence, according to the proof of the theorem the strict preference orders $A_3 \succ A_1 \succ A_2$ and $A_3 \succ A_2 \succ A_1$ cannot be achieved for any ω of decreasing nonnegative weights.

4 Open Problems

From the proof of the theorem it is clear that with three alternatives, there is at least one alternative which cannot be achieved as the winner or the loser by manipualting the weights. In example 1, A_2 cannot be achieved as the loser by manipulating the weights. In examples 2 and 3, A_3 cannot be achieved as the winner by manipulating the weights.

Is this in general true? That is, given any preference order profile is there always an alternative that cannot be achieved as a winner or a loser by manipulating the weights?

What are the preference order profiles for which A_1 is never the winner whatever weight system is taken?

It will be interesting to study the relationship between preference order profiles and the set of preference order relations that are not the societal preference orders for any weight system.

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