Economics Bulletin

Volume 35, Issue 3

A Dynamic Asset Pricing Model with Non-myopic Traders

Zhao Han
Department of Economics, Indiana University-Bloomington

Abstract

Dynamic asset pricing models built within the classic CARA-Normal framework usually assume myopic traders with one-period investment horizons or infinitely lived investors for tractability. I relax this myopic assumption and show the values of more finite trading opportunities are state-contingent and arise naturally as non-central chi^2-distributed. The moment generating function of the non-central chi^2 distribution thus can be utilized to derive the traders' first order conditions and preserve closed-form solutions. The model with non-myopic traders has a modified two-period overlapping generations(OLG) interpretation in which each young generation can have multiple investment opportunities.

I sincerely thank my advisor, Todd B. Walker, for many insightful discussions. All errors are my own.
Citation: Zhao Han, (2015) "A Dynamic Asset Pricing Model with Non-myopic Traders", *Economics Bulletin*, Volume 35, Issue 3, pages 1788-1794
Contact: Zhao Han - zhaohan@indiana.edu.
Submitted: June 21, 2015. **Published:** August 21, 2015.

1 Introduction

A large literature builds on the CARA-Normal (negative-exponential utility with Gaussian uncertainty) framework to explore interesting research questions related to various asset markets. Examples are ample and date back at least to Grossman and Stiglitz (1980). Valuable analytical tractability can usually be maintained by utilizing the moment generating function(m.g.f.) of the normal distribution.

However most applications either assume a two-period static environment or force the market to truncate in finite time; and models which allow the economy to be infinite-horizon often either impose myopic traders who only care single-period investment horizons or let the traders be infinitely lived. In this paper I relax the myopic assumption while keeping traders finite lifespan[1]. In particular, I show the values of more trading opportunities are state-contingent and arise naturally as non-central χ^2-distributed. Analyticity thus can be preserved by utilizing the m.g.f. of the non-central χ^2-distribution.

To the best of my knowledge, this finite non-myopic extension along with the non-central χ^2 characterization of the investment opportunities is novel. Campbell and Kyle (1993)[2] considers a similar model setup in continuous time and derives the investor's value function in terms of a quadratic form of normally distributed state variables. In contrast, the non-central χ^2-distributed investment opportunity raised in this paper is defined with respect to *each* additional investment horizon. Furthermore, investors in Campbell and Kyle (1993) are infinitely lived and their analytical result depends on noisy demand being an Ornstein-Uhlenbeck process. The model presented in the current paper is in discrete time and works for *any* covariance stationary ARMA dividend and noisy supply processes. There is a recent push to move away from the joint normality assumption made in CARA-Normal models[Breon-Drish (2015)] where the author explores a number of non-normal asset payoffs and supply in the static Grossman-Stiglitz framework. I contribute to the literature by showing how the non-central χ^2 distribution appears as a direct result of letting traders value longer investment horizons.

2 Model and Analysis

I consider a canonical CARA-Normal asset pricing model with stochastic dividend payments and noisy supply. The economy is infinite horizon. A representative trader, who cares N investment horizons, allocates wealth w_t between a perfect-elastically supplied risk-free asset(with constant gross return $\alpha = 1 + r > 1$) and a risky asset to maximize her expected CARA utility of terminal wealth $-E_t \exp(-\gamma w_{t+N})$. Both the dividend payment d_t and the supply s_t of the risky asset are stochastic and covariance stationary

$$\begin{aligned} d_t = & \ \overline{d} + D(L)\epsilon_t^d, \qquad \epsilon_t^d \sim N(0, \sigma_d^2); \\ s_t = & \ \overline{s} + S(L)\epsilon_t^s, \qquad \epsilon_t^s \sim N(0, \sigma_s^2); \end{aligned}$$

[1] The economy is still infinite-horizon though.

[2] Also see Wang (1993) for a continuous version model with infinitely lived investors.

The covariance stationarity of d_t and s_t imply both $D(L)$, $S(L)$ are (possibly) infinite-order square summable polynomials in the lag operator L. For example, an AR(1) d_t process

$$d_t - \overline{d} = \rho(d_{t-1} - \overline{d}) + \epsilon_t^d$$

implies $D(L) = \frac{1}{1-\rho L} = 1 + \rho L + \rho^2 L^2 + \rho^3 L^3 + ...$, thus satisfies the square-summability condition provided $|\rho| < 1$. The shocks $\{\epsilon_t^d, \epsilon_t^s\}$ are assumed orthogonal at all leads and lags. I consider a simple information environment where traders know the dividend and supply process $D(L)$ and $S(L)$ and observe the underlying shocks $\{\epsilon_t^d, \epsilon_t^s\}$ directly when they hit the economy. More complicated structure such as incomplete and asymmetric information will be left for future work.

The existing literature assumes $N = 1$ and interprets the above setup with a standard OLG structure where agents live for two periods and only consume when old. It is natural to allow traders to enjoy longer investment horizons[3]. One direct effect of allowing non-myopic($N > 1$ but finite) traders is now the investment horizons left will be a state variable. Let $x_{t,i}$ denote variable x's value at time t when the trader has i investment horizons left, a rational expectation equilibrium(REE) consists of a price system $\mathcal{P}_N = \{p_{t,1}, p_{t,2}, ..., p_{t,N}\}$ such that during any time, given $\mathcal{P}_N$, traders demand assets optimally and markets clear. The representative trader remembers the entire history of shocks $\{\epsilon_{t-j}^d, \epsilon_{t-j}^s\}_{j=0}^{\infty}$ and believes the equilibrium price process lies in the Hilbert space generated by them. This implies the equilibrium price is a linear function of the underlying shocks and covariance stationary, i.e.,

$$p_{t,i} = F_i(L)\epsilon_t^d + G_i(L)\epsilon_t^s.^4 \tag{1}$$

Consequently, $\mathcal{P}_N$ will be called linear and stationary if each element $p_{t,i}$ is linear and stationary. I start with the simplest non-myopic case $N = 2$. I solve the model by following the frequency domain approach in Walker and Whiteman (2007) and backward induction.

2.1 The $N = 2$ non-myopic case

Assume the representative trader (re-)enters the market at time t and let $V_{t+1,1}(w_{t+1,1})$ denote the trader's value function at time $t + 1$. Then

$$V_{t+1,1}(w_{t+1,1}) = \max_{z_{t+1,1}} -E_{t+1} \exp(-\gamma w_{t+2,0}),$$

$$\text{s.t}: \quad w_{t+2,0} = z_{t+1,1}(p_{t+2,2} + d_{t+2}) + \alpha(w_{t+1,1} - z_{t+1,1}p_{t+1,1}).$$

Notice that at $t + 2$ investment horizons left will become 2 again due to the trader's re-entrance into the market. The linearity of

$$p_{t,2} = F_2(L)\epsilon_t^d + G_2(L)\epsilon_t^s; \tag{2}$$

$$p_{t,1} = F_1(L)\epsilon_t^d + G_1(L)\epsilon_t^s. \tag{3}$$

[3] Albagli (2015) considers the standard N-period OLG setup and discusses the risk sharing effects in an asymmetric information environment for both AR(1) dividend and supply processes. To isolate these effects, I instead maintain the representative trader assumption and only relax the "myopic" part. The technique advocated here can be applied to this OLG environment as well.

[4] From now on we normalize $\overline{d} = 0, \overline{s} = 0$, which implies $\overline{p}_i = 0$. Consequently the derived equilibrium price $p_{t,i}$ has an interpretation of deviations from its steady state.

implies $w_{t+2,0}$ is normally distributed conditional on time $t+1$. Applying the normal m.g.f. to $-E_{t+1}\exp(-\gamma w_{t+2,0})$, the trader's demand function follows from the first-order necessary condition for maximization and is given by

$$z^*_{t+1,1} = \frac{1}{\gamma \mathrm{var}_{t+1}(p_{t+2,2}+d_{t+2})}[E_{t+1}(p_{t+2,2}+d_{t+2}) - \alpha p_{t+1,1}]; \tag{4}$$

which is a classic result where the demand function is optimized over the mean and variance of excess returns. Market clearing requires $z^*_{t+1,1} \equiv s_{t+1} = S(L)\epsilon^s_{t+1}$. Plugging the prices forms (2), (3) along with market clearing condition yield a set of equilibrium conditions $F_i(z), G_i(z), i = 1,2$ have to satisfy

$$z^{-1}[F_2(z) - F_2(0)] + z^{-1}[D(z) - D(0)] = \alpha F_1(z), \tag{5}$$

$$z^{-1}[G_2(z) - G_2(0)] - \alpha G_1(z) = \gamma[(F_2(0) + D(0))^2\sigma_d^2 + G_2(0)^2\sigma_s^2]S(z). \tag{6}$$

This in turn determines [5] ,

$$V_{t+1,1}(w_{t+1,1}) = -\exp\{-\gamma\big[\alpha w_{t+1,1} + \frac{1}{2}\gamma \mathrm{var}_{t+1}(p_{t+2,2}+d_{t+2})s^2_{t+1}\big]\}. \tag{7}$$

While $\alpha w_{t+1,1}$ represents the "time" value of this investment horizon in which a risk-free return α is guaranteed, $\frac{1}{2}\gamma\mathrm{var}_{t+1}(p_{t+2,2}+d_{t+2})s^2_{t+1}$ represents the value of the investment opportunity: It is non-negative and larger risk aversion γ indicates a higher value. Interestingly, higher conditional variance $\mathrm{var}_{t+1}(p_{t+2,2}+d_{t+2})$ induces a higher investment opportunity value. I argue it is due to the stronger hedging effect it could bring to traders. Due to the two-period investment horizons, investors will not operate on mean-variance frontier during the first period and an inter-temporal hedging demand component has to be taking into consideration. For instance, the traders *might* be willing to hold more risky asset than the amount implied by a typical mean-variance demand of the form (4). The investors are willing to do so if they expect the risky asset's performance will be lackluster in the first period but could bounce back in the second period due to the larger variance.

Finally, the investment opportunity value is state-contingent, depending on realized noisy supply squared s^2_{t+1}. This is due to the no short sale constraint the model structure implicitly assumed. As long as there is noisy supply(or demand[6]), prices will fluctuate and the traders can always take long or short positions to take advantage of this investment opportunity. Conditional on time t, s^2_{t+1} is the only stochastic component in the investment opportunity value[7] and causes the value to be non-central χ^2-distributed: while the χ^2 attribute comes from the normal variable s_{t+1} squared, the non-central attribute originates from the potential persistence the noisy supply process $S(L)$ could have.

Consequently, at time t, the trader solves

$$V_{t,2}(w_{t,2}) = \max_{z_{t,2}} E_t V_{t+1,1}(w_{t+1,1}),$$

[5] $V_{t+1,1}(w_{t+1,1}) = -\exp\left(-\gamma E_{t+1}w^*_{t+2,0} + \frac{1}{2}\gamma^2\mathrm{var}_{t+1}(w^*_{t+2,0})\right) = -\exp\left(-\gamma[\alpha w_{t+1,1} + z^*_{t+1,1}(E_{t+1}(p_{t+2,2} + d_{t+2}) - \alpha p_{t+1,1})] + \frac{1}{2}\gamma^2\mathrm{var}_{t+1}(p_{t+2,2}+d_{t+2})(z^*_{t+1,1})^2\right) = -\exp\{-\gamma\big[\alpha w_{t+1,1} + \frac{1}{2}\gamma\mathrm{var}_{t+1}(p_{t+2,2}+d_{t+2})(z^*_{t+1,1})^2\big]\}$ where I use the normal m.g.f. in the first equality, combine the budget constraint in the second one, and utilize the first-order condition (4) in the last equality.

[6] When $s_t < 0$, traders become the net supplier of the risky asset.

[7] $\mathrm{var}_{t+1}(p_{t+2,2}+d_{t+2}) = (F_2(0)+D(0))^2\sigma_d^2 + G_2(0)^2\sigma_s^2$ is a constant by construction;

$$\text{s.t}: \; w_{t+1,1} = z_{t,2}(p_{t+1,1} + d_{t+1}) + \alpha(w_{t,2} - z_{t,2}p_{t,2}).$$

Define $A_t = L^{-1}[S(L) - S_0]\epsilon_t^s$, $B_t = -\gamma\alpha$, and $C_t = -\frac{1}{2}\gamma^2(F_2(0) + D(0))^2\sigma_d^2 + G_2(0)^2\sigma_s^2)$, A_t, B_t and C_t are all constants conditional on time t[8] and

$$V_{t+1,1}(w_{t+1,1}) = -\exp(C_t A_t^2)\exp(B_t w_{t+1,1} + C_t S_0^2(\epsilon_{t+1}^s)^2 + 2C_t A_t S_0 \epsilon_{t+1}^s). \tag{8}$$

Thus, (8) depends on the realization of the supply shock squared $(\epsilon_{t+1}^s)^2$ given $S_0 \neq 0$.[9]

Combining the budget constraint along with the prices forms (2), (3) and using the assumption $\{\epsilon_t^d, \epsilon_t^s\}$ are orthogonal at all leads and lags,

$$\begin{aligned} V_{t,2}(w_{t,2}) = \max_{z_{t,2}} -\exp(C_t A_t^2)\exp(\alpha B w_{t,2})\exp\big(-\alpha B_t z_{t,2}[F_2(L)\epsilon_t^d + G_2(L)\epsilon_t^s]\big) \\ \exp(B_t z_{t,2} L^{-1}[G_1(L) - G_1(0)]\epsilon_t^s) E_t \exp(B_t z_{t,2}(F_1(L) + D(L))\epsilon_{t+1}^d) \\ E_t \exp(C_t S_0^2(\epsilon_{t+1}^s)^2 + (B_t z_{t,2} G_0 + 2C_t A_t S_0)\epsilon_{t+1}^s). \end{aligned} \tag{9}$$

The normal m.g.f. and the Wiener-Kolmogorov formula imply

$$\begin{aligned} E_t \exp(B_t z_{t,2}(F_1(L) + D(L))\epsilon_{t+1}^d) = \exp(\frac{1}{2}B_t^2 z_{t,2}^2(F_1(0) + D(0))^2\sigma_d^2) \\ \exp(B_t z_{t,2} L^{-1}[F_1(L) + D(L) - F_1(0) - D(0)]\epsilon_t^d). \end{aligned} \tag{10}$$

Completing the squares of $C_t S_0^2(\epsilon_{t+1}^s)^2 + (B_t z_{t,2} G_0 + 2C_t A_t S_0)\epsilon_{t+1}^s$ in the second expectation term of (9) yields

$$\begin{aligned} E_t \exp(C_t S_0^2(\epsilon_{t+1}^s)^2 + (B_t z_{t,2} G_0 + 2C_t A_t S_0)\epsilon_{t+1}^s) = \\ \exp(-\frac{(B_t z_t G_0 + 2C_t A_t S_0)^2}{4C_t S_0^2}) E_t \exp(C_t S_0^2(\epsilon_{t+1}^s + \frac{B_t z_t G_0 + 2C_t A_t S_0}{2C_t S_0^2})^2). \end{aligned}$$

Since $\epsilon_{t+1}^s + \frac{B_t z_t G_0 + 2C_t A_t S_0}{2C_t S_0^2} \sim N(\frac{B_t z_t G_0 + 2C_t A_t S_0}{2C_t S_0^2}, \sigma_s^2)$, $\big(\frac{\epsilon_{t+1}^s + \frac{B_t z_t G_0 + 2C_t A_t S_0}{2C_t S_0^2}}{\sigma_s}\big)^2$ follows a non-central χ-square distribution with degree of freedom 1 and the non-centrality parameter $\big(\frac{B_t z_t G_0 + 2C_t A_t S_0}{2C_t S_0^2 \sigma_s}\big)^2$. Applying the m.g.f. of the non-central χ-squared distribution[10] gives

$$\begin{aligned} E_t \exp(C_t S_0^2(\epsilon_{t+1}^s)^2 + B_t z_t G_0 \epsilon_{t+1}^s + 2C_t A_t S_0 \epsilon_{t+1}^s) = \\ (1 - 2C_t S_0^2 \sigma_s^2)^{-\frac{1}{2}} \exp\big(\frac{\sigma_s^2(B_t z_t G_0 + 2C_t A_t S_0)^2}{2 - 4C_t S_0^2 \sigma_s^2}\big). \end{aligned} \tag{11}$$

Plugging (10),(11) along with $A_t = L^{-1}[S(L) - S_0]\epsilon_t^s$ into (9) and maximizing $V_{t,2}(w_t)$ with respect to the choice variable $z_{t,2}$ yields the trader's first order condition. Imposing the

[8] $A_t = \sum_{j=1}^{\infty} S_j \epsilon_{t+1-j}^s$; $C_t = -\frac{1}{2}\gamma^2 \mathrm{var}_{t+1}(p_{t+2,2} + d_{t+2})$.

[9] $S_0 = 0$ implies traders have one-period foresight of the noisy supply, under this scenario the normal m.g.f. technique is still valid.

[10] If $X \sim \chi^2(r, \lambda)$, where r is the degree of freedom and λ is the non-centrality parameter, then $M_X(t) = Ee^{tX} = (1 - 2t)^{-r/2}\exp\{\frac{\lambda t}{1-2t}\}$ for $1 - 2t > 0$.

market clearing condition $z_{t,2} = S(L)\epsilon_t^s$ yields another set of equilibrium conditions,

$$z^{-1}[F_1(z) - F_1(0)] + z^{-1}[D(z) - D(0)] = \alpha F_2(z), \tag{12}$$

$$z^{-1}[G_1(z) - G_1(0)] - \alpha G_2(z) = -B_t(F_1(0) + D(0))^2\sigma_d^2 S(z) - \frac{B_t\sigma_s^2 G_1(0)^2}{1 - 2C_t S_0^2\sigma_s^2}S(z) - \frac{2G_1(0)C_t S_0\sigma_s^2}{1 - 2C_t S_0^2\sigma_s^2}z^{-1}[S(z) - S_0]. \tag{13}$$

Combining (5) and (12) and solving $F_1(L)$ in terms of $F_1(0)$, $F_2(0)$ and $D(L)$ leads to a unique set of $\{F_1^*(z), F_2^*(z)\}$

$$\begin{aligned} F_1^*(z) &= F_2^*(z) = \frac{D(\alpha^{-1}) - D(z)}{1 - \alpha z}, \\ F_1^*(0) &= F_2^*(0) = D(\alpha^{-1}) - D(0); \end{aligned}$$

Combining (6) and (13) and letting $M(z)$, $N(z)$ denote the right hand side of (6), (13) give

$$\begin{aligned} G_1(z) &= \frac{G_1(0) + \alpha z G_2(0) + zN(z) + \alpha z^2 M(z)}{1 - \alpha^2 z^2}, \\ G_2(z) &= \frac{G_2(0) + \alpha z G_1(0) + zM(z) + \alpha z^2 N(z)}{1 - \alpha^2 z^2}. \end{aligned}$$

Covariance stationarity of $G_i(L)$ indicates analyticity of $G_i(z)$ in the open unit disk $|z| < 1$, which implies $G_1(0)$, $G_2(0)$ must be set to remove the poles at $z = \pm\alpha^{-1}$[See Whiteman (1983)], thus

$$G_1(0) + G_2(0) + \alpha^{-1}[N(\alpha^{-1}) + M(\alpha^{-1})] = 0; \tag{14}$$

$$G_1(0) - G_2(0) - \alpha^{-1}[N(-\alpha^{-1}) + M(-\alpha^{-1})] = 0. \tag{15}$$

The above equations are nonlinear since $G_2(0)^2$ appears in $M(z)$ while $G_1(0)^2$ along with $G_1(0)$ and $G_2(0)^2$ terms appears in $N(z)$[11]. While it is solely the conditional variance $\text{var}_{t+1}(p_{t+2,2} + d_{t+2}) = D(\alpha^{-1})^2\sigma_d^2 + G_2(0)^2\sigma_s^2$ that introduces such nonlinearity in $M(z)$ when the trader has to liquidate her portfolio, it is the combination of the agent's concerns about the volatilities $\text{var}_{t+1}(p_{t+2,2} + d_{t+2}), \text{var}_t(p_{t+1,1} + d_{t+1})$ and her hedging motives that introduce more complicated nonlinearities in $N(z)$. To see this more explicitly, $N(z)$ can be written as

$$\begin{aligned} N(z) &= \frac{\gamma^2 S_0\sigma_s^2 G_1(0)\text{var}_{t+1}(p_{t+1,2} + d_{t+2})}{1 + \gamma^2 S_0^2\sigma_s^2\text{var}_{t+1}(p_{t+1,2} + d_{t+2})}[z^{-1}(S(z) - S_0) - \\ &\quad \gamma\alpha S_0 G_1(0)S(z)] + \gamma\alpha\text{var}_t(p_{t+1,1} + d_{t+1}); \end{aligned}$$

The mixed effects of trader's concerns on market volatility and her hedging motives can be seen as $\text{var}_{t+1}(p_{t+1,2} + d_{t+2})$ appearing both in the denominator and the numerator of the first term in the above expression. Under what scenarios one effect dominates the other is an interesting question and deserves future investigation.

[11]recall that in (13) C_t involves $G_2(0)^2$;

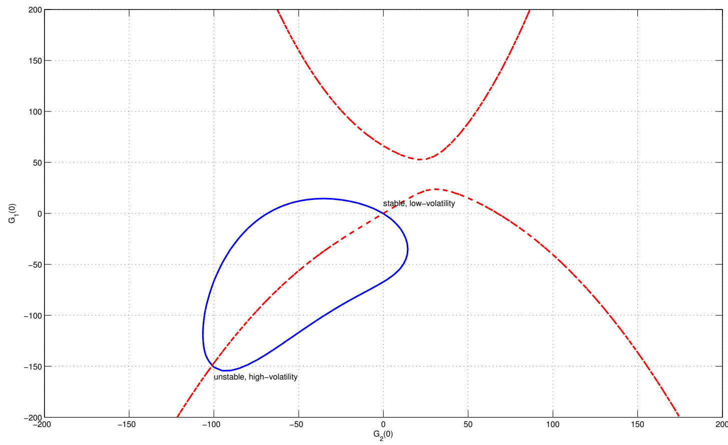

Figure 1: **Two Equilibria with Non-Myopic Traders**: This figure plots solution sets of the two non-linear equations specified in (14), (15). Parameter values: $N = 2, r = 0.02, \gamma = 1.5, \sigma_d = 0.03, \sigma_s = 0.1$ with $d_t = 0.9d_{t-1} + \epsilon_t^d$, $s_t = -0.01s_{t-1} + \epsilon_t^s$. There are two intersections and each defines an equilibrium price: $\{G_1^*(0), G_2^*(0)\} = \{-0.095, -0.097\}$ or {-149.427, -101.104 }.

A nonlinear solver indicates there are two sets of solutions $\{G_1^*(0), G_2^*(0)\}$ satisfying the equations (14), (15) thus we will have two sets of $\{G_1^*(z), G_2^*(z)\}$, which determines two equilibria $\mathcal{P}_2$. The multiplicity result resonates with the myopic model. Furthermore, the equilibria can be characterized as a stable low-volatility price, and a unstable high-volatility price [Walker and Whiteman (2007), Albagli (2015)]. For an illustrative example, see Figure 1. A calculation of $V_{t,2}(w_{t,2})$ implies the above backward induction procedure can be continued to allow more investment horizons($N > 2$). I leave the general N case to future work.

The non-myopic cases have a modified two-period OLG interpretation in which traders will have N opportunities to refresh their portfolios during young. This interpretation is convenient in explaining how the underlying demographics shapes the investors' beliefs: Traders are sure prices will transit naturally from $p_{t,N}$ to $p_{t+1,N-1}$ to $p_{t+2,N-2}$...during their investment horizons; and when they have to liquidate and quit the market(either temporarily or permanently), they also know the new generation will hold the same belief such that prices in the next period will start from $p_{t+N,N}$ again. Overtime, prices of the risky asset will display "cyclical" patterns and time-varying conditional variances. The price system $\mathcal{P}_N$, however, is still stationary. While the CARA-Normal assumptions preserves the linearity of prices, the stationary demographics behind the OLG structure ensures prices in the future will behave in similar pattens as the past and today's, which provides a necessary force to anchor the forward-looking traders' expectations and guarantees the stationarity of the price system in such dynamic settings.

3 Conclusion

In a canonical dynamic CARA-Normal asset pricing framework, I relax the myopic assumption and solve the model analytically by showing the values of trading opportunities are non-central χ^2 distributed. More investment horizons introduce complicated nonlinear mixed effects of traders' concerns on market volatilities and their hedging motives. The analytical results derived for general dividend and supply processes are valuable for topics like persistence liquidity trading and asymmetric information.

References

Albagli, E. (2015). Investment horizons and asset prices under asymmetric information. *Journal of Economic Theory*, 158, Part B(0):787 – 837. Symposium on Information, Coordination, and Market Frictions.

Breon-Drish, B. (2015). On existence and uniqueness of equilibrium in a class of noisy rational expectations models. *Review of Economic Studies*.

Campbell, J. Y. and Kyle, A. S. (1993). Smart money, noise trading and stock price behaviour. *The Review of Economic Studies*, 60(1):pp. 1–34.

Grossman, S. J. and Stiglitz, J. E. (1980). On the Impossibility of Informationally Efficient Markets. *American Economic Review*, 70(3):393–408.

Walker, T. B. and Whiteman, C. H. (2007). Multiple equilibria in a simple asset pricing model. *Economics Letters*, 97(3):191 – 196.

Wang, J. (1993). A Model of Intertemporal Asset Prices under Asymmetric Information. *Review of Economic Studies*, 60(2):249–82.

Whiteman, C. H. (1983). *Linear Rational Expectations Models*. University of Minnesota Press.