Economics Bulletin

Volume 34, Issue 2

A contest success function with a rent-dependent dissipation rate

Normann Lorenz
University of Trier

Abstract

In this note a new contest success function is derived that results in investments in equilibrium that are proportional to the square of the rent, which implies a dissipation rate that is not independent of, but increasing in the rent. Only for a contest success function with this property can it be optimal to use least squares regression models to determine transfers for a risk adjustment scheme (a regulatory means to reduce risk selection in health insurance markets). A second property of this new contest success function is that winning probabilities do not depend only on the ratio or only on the difference of investments, but on both the ratio and the difference; this may make it more suitable than common contest success functions for some situations like corruption or getting tenure at a university.

Universitaet Trier, Universitaetsring 15, 54286 Trier, Germany; e-mail: lorenzn@uni-trier.de; phone: 0049-651-2012624. I thank Friedrich Breyer, Mathias Kifmann, Dominik Sachs and two anonymous referees for helpful comments and suggestions.
Citation: Normann Lorenz, (2014) "A contest success function with a rent-dependent dissipation rate", *Economics Bulletin*, Vol. 34 No. 2 pp. 1091-1102.
Contact: Normann Lorenz - lorenzn@uni-trier.de.
Submitted: January 21, 2014. **Published:** May 25, 2014.

1 Introduction

There exists by now a huge literature on the theory of contests that has been applied to fields as diverse as rent-seeking (Tullock 1980), the legal system (Baye et al. 2005), conflict situations (Garfinkel and Skaperdas 2007) and sports (Szymanski 2003).[1]

The theory of contests can also be applied to the problem of risk selection that arises in regulated health insurance markets if insurers are required to charge a uniform premium for all insured, irrespective of their risk type.[2] Because the expected cost of the low risk individuals is below the uniform premium, insurers will compete for the (positive) rents accruing from these individuals. In a similar vein, because the expected cost of the high risk individuals is above the premium, insurers will spend resources to avoid having to bear the (negative) rents associated with these individuals. Insurers' investments in such a risk selection contest are a waste of resources. Therefore, regulators often try to reduce such investments by implementing a risk adjustment scheme, a transfer system that imposes taxes on and pays subsidies to health insurers depending on their risk structure. These transfers are usually determined as the predicted values of a least squares regression. The least squares regression, of course, minimizes the sum of squared residuals. Because these residuals are equal to the (positive and negative) rents, this would only be optimal if investments in the risk selection contest were proportional to the square of the rent.[3]

Investments that are proportional to the square of the rent imply a dissipation rate (sum of investments as a share of the rent) that is increasing in, and is in fact proportional to the rent. This stands in sharp contrast to the contest literature, where such a property of the dissipation rate has not yet been described. Although there has been a long debate on whether a dissipation rate that differs from one can occur in a rent-seeking situation, basically all models predict a constant dissipation rate, i.e. a dissipation rate that does not depend on the value of the rent (see the collection of papers on this topic by Lockard and Tullock (2001)).

It is therefore the purpose of this note to analyze whether there exists a contest success function (c.s.f.) which induces investments in equilibrium that are proportional to the square of the rent. As we show in the following Section 2, such a c.s.f. does indeed exist.

Two additional properties of this c.s.f. are discussed in Section 3: First, winning probabilities do not depend only on the ratio (as for the Tullock-c.s.f.) or only on the difference of investments (as for the Hirshleifer-c.s.f.), but on both the ratio *and* the difference: Increasing the difference while holding the ratio constant increases high and decreases low winning probabilities. The same holds for increasing the ratio while holding the difference constant. We think that this property may be a more appropriate description of a large number of situations, among them corruption or getting tenure at a university. A second additional property of this c.s.f. is that the sum of investments is first increasing and then decreasing in the number of players. This

[1]For an overview see the survey by Konrad (2009) and the collection of papers by Congleton et al. (2008).

[2]For the problem of risk selection see van de Ven and Ellis (2000); for an application of contest theory to the problem of risk selection see Lorenz (2014).

[3]For a more detailed exposition of this argument see Lorenz (2014).

would be important for the optimal design of a contest when aggregate investments are to be minimized or maximized. Section 4 concludes.

2 The contest success function

2.1 Two players

Consider a situation with two risk-neutral players A and B who compete for a (positive) rent D.[4] They invest amounts a and b and win the rent with probability $p(a,b)$ and $1-p(a,b)$, respectively. Following the axiomatization by Skaperdas (1996), we consider the c.s.f. to be of the logistic form:

$$p(a,b)=\frac{g(a)}{g(a)+g(b)}. \tag{1}$$

For this class of c.s.f. we can state the following proposition:

Proposition 1. *The only contest success function* $p(a,b)=\frac{g(a)}{g(a)+g(b)}$ *that induces investments in a symmetric pure strategy equilibrium which are proportional to the square of the rent is*

$$p(a,b)=\frac{e^{m\sqrt{a}}}{e^{m\sqrt{a}}+e^{m\sqrt{b}}} \quad \textit{with} \quad 0<m<5.49D^{-\frac{1}{2}}. \tag{2}$$

Proof. With this c.s.f., the optimization problem of player A is

$$\max_a \ \pi^A=p(a,b)D-a=\frac{e^{m\sqrt{a}}}{e^{m\sqrt{a}}+e^{m\sqrt{b}}}D-a,$$

and likewise for player B. The first order conditions are

$$\frac{\partial \pi^A}{\partial a} = \frac{\frac{m}{2\sqrt{a}}e^{m\sqrt{a}}e^{m\sqrt{b}}}{(e^{m\sqrt{a}}+e^{m\sqrt{b}})^2}D-1=0 \tag{3}$$

$$\frac{\partial \pi^B}{\partial b} = \frac{\frac{m}{2\sqrt{b}}e^{m\sqrt{a}}e^{m\sqrt{b}}}{(e^{m\sqrt{a}}+e^{m\sqrt{b}})^2}D-1=0. \tag{4}$$

These two equations imply $a=b$, so the equilibrium is symmetric. Inserting $a=b$ into (3) and (4) yields the unique Nash-equilibrium

$$a^*=\frac{m^2D^2}{64}=b^*,$$

so that investments are indeed proportional to the square of the rent.

It is now shown that (2) is the only c.s.f. with this property. Profit for insurer A equals

$$\pi^A=p(a,b)D-a=\frac{g(a)}{g(a)+g(b)}D-a,$$

[4] The case of a negative rent, which insurers aim not to win, is analogous.

with the first order condition

$$\frac{\partial \pi^A}{\partial a} = \frac{g'(a)g(b)}{(g(a)+g(b))^2}D - 1 = 0.$$

In the symmetric equilibrium $a^* = b^*$, so this first order condition can be simplified to

$$\frac{g'(a)}{4g(a)}D - 1 = 0. \tag{5}$$

Since a^* is proportional to the square of the rent, we have $a^* = sD^2$, or $D = s^{-\frac{1}{2}}a^{\frac{1}{2}}$. Inserting into (5) results in

$$g'(a) = 4s^{\frac{1}{2}}a^{-\frac{1}{2}}g(a).$$

This differential equation can be solved to yield

$$g(a) = ke^{\int 4s^{\frac{1}{2}}a^{-\frac{1}{2}}da} = ke^{8\sqrt{s}\sqrt{a}} = ke^{m\sqrt{a}}.$$

The contest success function therefore is:

$$p(a,b) = \frac{g(a)}{g(a)+g(b)} = \frac{ke^{m\sqrt{a}}}{ke^{m\sqrt{a}} + ke^{m\sqrt{b}}} = \frac{e^{m\sqrt{a}}}{e^{m\sqrt{a}} + e^{m\sqrt{b}}}.$$

The analytical derivation of the sufficient condition $m < 4.9$ for the existence of a pure strategy Nash-equilibrium can be found in Appendix A1, and the numerical derivation of the necessary and sufficient condition $m < 5.49$ can be found in Appendix A2. □

2.2 Extension to n players

The contest success function (2) can easily be extended to the case of n players:

$$p(a_i, a_{-i}) = \frac{e^{m\sqrt{a_i}}}{e^{m\sqrt{a_i}} + \sum_{\substack{j=1 \\ j\neq i}}^{n} e^{m\sqrt{a_j}}}.$$

Here, a_i denotes the investment of player i and a_{-i} the vector of investments of all the other players. The unique Nash-equilibrium is

$$a_i^* = \frac{m^2(n-1)^2D^2}{4n^4},$$

so investments are proportional to the square of the rent. Individual investments are decreasing in the number of players, but aggregate investments are not monotone in n, see Section 3.2. The conditions for the existence of this pure strategy Nash-equilibrium can be found in Appendix A3.

3 Additional properties of the contest success functions

3.1 Winning probabilities depend on the ratio and the difference of investments

Hirshleifer (1989) suggested to distinguish contest success functions according to whether winning probabilities depend on the ratio or on the difference of investments. He showed that the Tullock-c.s.f., $p(a,b) = \frac{a^r}{a^r+b^r}$, belongs to the first category, while the Hirshleifer-c.s.f., $p(a,b) = \frac{e^a}{e^a+e^b}$, belongs to the second. The contest success function (2) does not belong to either of the two categories; instead, winning probabilities depend on both the ratio and the difference of investments. Increasing the difference while holding the ratio constant (by multiplying all investments by a factor $t > 1$) increases high and decreases low winning probabilities:

Proposition 2. *For the contest success function (2), if $a > b$, so that $p(a,b) > \frac{1}{2}$, then $p(ta,tb) > p(a,b)$ for $t > 1$.*

Proof. Assume $a > b$; then

$$\frac{\partial p(ta,tb)}{\partial t} = \frac{me^{m\sqrt{at}}e^{m\sqrt{bt}}}{2\sqrt{t}\left(e^{m\sqrt{at}}+e^{m\sqrt{bt}}\right)^2}\left(\sqrt{a}-\sqrt{b}\right) > 0.$$

□

On the other hand, increasing the ratio while holding the difference constant (by reducing all investments by the same amount $t > 0$) increases high and decreases low winning probabilities as well:

Proposition 3. *For the contest success function (2), if $a > b$, so that $p(a,b) > \frac{1}{2}$, then $p(a-t,b-t) > p(a,b)$ for $0 < t \leq b$.*

Proof. Assume $a > b$; then

$$\frac{\partial p(a-t,b-t)}{\partial t} = \frac{me^{m\sqrt{a-t}}e^{m\sqrt{b-t}}}{2\left(e^{m\sqrt{a-t}}+e^{m\sqrt{b-t}}\right)^2}\left(\frac{1}{\sqrt{b-t}}-\frac{1}{\sqrt{a-t}}\right) > 0.$$

□

We think that with the properties stated in Proposition 2 and 3, the c.s.f. given by (2) may provide a more suitable description of a number of conflict situations. Consider, e.g., the case of corruption: It seems quite plausible that the person deciding on the distribution of the rent is more susceptible to a twice as large bribe from one of the rent seekers when twice as large implies a large absolute difference, e.g. \$1000 vs. \$2000 compared to \$5 vs. \$10. Similarly, he may also be more susceptible to the higher of a pair of bribes that differ by some absolute amount if this implies a larger difference in relative terms, e.g. \$10 vs. \$0 compared to \$110 vs. \$100.

Or consider the case of two assistant professors being employed by the same department competing for tenure (or applying for the position of assistant professor in the first place) which only one of them will get: If one of them published one paper in an A$^+$-journal while the other did not, he will be the one getting tenure almost for sure. If one of them published five papers and the other published four, the case is not clear-cut any more, and the one with the smaller number of A^+-publications may have a considerable chance, too.

Another example is political campaigning: A politician spending, say, a million dollars on his campaign against an opponent who does not campaign at all will win the election with a higher probability than if he spends, say, five million and his opponent spends four. In the first case voters do not know the second candidate at all, while in the second they may know both candidates almost equally well.

One can come up with numerous further examples; in fact, it is quite difficult to imagine any contest where winning probabilities should indeed depend only on the ratio or only on the difference, and not on both as implied by the c.s.f. (2).

3.2 Aggregate investments are non-monotone in the number of players

A second property of the contest success function derived in Section 2 is that the sum of investments is not monotonically increasing or decreasing in the number of players, n. Rather, it is first increasing and then decreasing, and is maximal for $n = 3$.

Proposition 4. *For the contest success function*

$$p(a_i, a_{-i}) = \frac{e^{m\sqrt{a_i}}}{e^{m\sqrt{a_i}} + \sum_{\substack{j=1 \\ j\neq i}}^{n} e^{m\sqrt{a_j}}}$$

the sum of investments is maximal for $n = 3$, and decreasing for $n > 3$.

Proof. Aggregate investments are

$$\sum_{i=1}^{n} a_i^* = \frac{m^2 D^2 (n-1)^2}{4n^3}.$$

The first order condition with respect to n is

$$\frac{\partial \sum_{i=1}^{n} a_i^*}{\partial n} = \frac{-(n-1)(n-3)}{n^4} \frac{m^2 D^2}{4}$$

so that we have

$$\frac{\partial \sum_{i=1}^{n} a_i^*}{\partial n} \gtreqqless 0 \quad \text{for} \quad \begin{cases} 2 \leq n < 3 \\ n = 3 \\ n > 3 \end{cases}.$$

□

The number of players, ordered according to decreasing aggregate investments, is 3, 4, 5, 2, 6, 7, 8, ... If a contest designer wants to minimize aggregate investments (as in the case of corruption), he should have as many players as possible, unless the number of players can not be larger than five; only in this case should there be two players. If he wants to maximize aggregate investments he should design a contest with three players.

4 Conclusion

The aim of this note was to determine whether there exists a contest success function that induces investments in equilibrium that are proportional to the square of the rent. A contest success function with this property was derived and two additional properties were discussed: First, the winning probabilities depend on both the ratio and the difference of investments. Secondly, the sum of investments is not monotone in the number of players; instead it is maximized for three players. It was argued that this c.s.f. may be more suitable for a large number of situations, among them corruption or getting tenure at a university.

For future research it would be interesting to examine whether the main property of this c.s.f. – a dissipation rate that increases in the value of the rent – can be shown to exist with a formal empirical test.

Appendix

Appendix A1

In this section a sufficient condition for the existence of an equilibrium in pure strategies is derived, the concavity of $\pi(a, b^*)$. We have

$$\begin{aligned}
\pi^A(a, b^*) &= \frac{e^{m\sqrt{a}}}{e^{m\sqrt{a}} + e^{m\sqrt{b^*}}} D - a \\
\frac{\partial \pi^A(a, b^*)}{\partial a} &= \frac{e^{m\sqrt{a}}}{\sqrt{a}(e^{m\sqrt{a}} + e^{m\sqrt{b^*}})^2} \frac{me^{m\sqrt{b^*}}}{2} D - 1 = 0 \\
\frac{\partial^2 \pi^A(a, b^*)}{\partial a^2} &= \frac{me^{m(\sqrt{a}+\sqrt{b^*})}}{4a^{\frac{3}{2}}(e^{m\sqrt{a}} + e^{m\sqrt{b^*}})^3} D \left[(m\sqrt{a} - 1)e^{m\sqrt{b^*}} - (m\sqrt{a} + 1)e^{m\sqrt{a}}\right].
\end{aligned}$$

The sign of the second order condition only depends on

$$\left[(m\sqrt{a} - 1)e^{m\sqrt{b^*}} - (m\sqrt{a} + 1)e^{m\sqrt{a}}\right]. \tag{6}$$

To simplify the notation, let $m\sqrt{a} = x$ and $m\sqrt{b^*} = y$ and denote (6) by $f(x, y)$:

$$f(x, y) = (x - 1)e^y - (x + 1)e^x.$$

For $\pi^A(a, b^*)$ to be strictly concave in a, $f(x, y)$ has to be negative. To derive this condition, two cases $x \leq 1$ and $x > 1$ have to be distinguished:

Case 1: $x \leq 1$

For $x \leq 1$, $f(x, y)$ is negative for all y:

$$f(x, y) = (x - 1)e^y - (x + 1)e^x < 0 \quad \forall\, y,\ x \leq 1.$$

Case 2: $x > 1$

One can first show that $f(x, y)$ is concave in x:

$$\frac{\partial^2 f}{\partial x^2} = -(x + 3)e^x < 0 \quad \forall\, y.$$

In addition, $f(x, y)$ is negative for $x = 0$:

$$f(0, y) = -e^y - 1 < 0 \quad \forall\, y.$$

Now, the following two cases, $y \leq \ln 2$ and $y > \ln 2$, have to be distinguished:

Case 2a): $x > 1$ and $y \leq \ln 2$

For $y \leq \ln 2$, the derivative of $f(x, y)$ with respect to x for $x = 0$ is:

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = e^y - 2 \leq 0 \quad \forall\, y \leq \ln 2.$$

With $f(0, y) < 0$, $\left.\frac{\partial f}{\partial x}\right|_{x=0} \leq 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$, it holds that

$$f(x, y) < 0 \quad \forall \quad y \leq \ln 2,\ x > 1.$$

Case 2b): $x > 1$ and $y > \ln 2$

For $y > \ln 2$, the derivative of $f(x, y)$ with respect to x for $x = 0$ is:

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = e^y - 2 > 0 \quad \forall\, y > \ln 2.$$

The derivative for $x = y$ is:

$$\left.\frac{\partial f}{\partial x}\right|_{x=y} = -(1 + y)e^y < 0 \quad \forall\, y > \ln 2.$$

With $\left.\frac{\partial f}{\partial x}\right|_{x=0} > 0$, $\left.\frac{\partial f}{\partial x}\right|_{x=y} < 0$, $\frac{\partial^2 f}{\partial x^2} < 0$, and because f is continuously differentiable, $f(x, y)$ has a maximum for x in $[0, y]$, given y. For $\pi^A(a, b^*)$ to be strictly concave, this maximum must be negative.

For $x > 1$, $f(x, y)$ is increasing in y:

$$\frac{\partial f}{\partial y} = (x - 1)e^y > 0 \quad \forall\, x > 1.$$

Therefore, there exists a $\bar{y} > \ln 2$, for which the maximum of $f(x, \bar{y})$ with respect to x is equal to zero. For this $\bar{y}$ it holds that:

$$\begin{aligned} f(x, \bar{y}) &= 0 \\ \frac{\partial f(x, \bar{y})}{\partial x} &= 0. \end{aligned}$$

Inserting $f(x, \bar{y})$ and $\frac{\partial f(x,\bar{y})}{\partial x}$ yields

$$(x - 1)e^{\bar{y}} - (x + 1)e^x = 0 \qquad (7)$$

$$e^{\bar{y}} - (x + 2)e^x = 0. \qquad (8)$$

Solving (8) for $e^{\bar{y}}$ and substituting in (7), we have

$$(x^2 - 3)e^x = 0,$$

which yields $x = \sqrt{3}$. Inserting into (8) and solving for $\bar{y}$ yields

$$\bar{y} = \ln\left((\sqrt{3}+2)e^{\sqrt{3}}\right) = \sqrt{3} + \ln(\sqrt{3}+2).$$

We therefore have

$$\frac{m^2 D}{8} < \sqrt{3} + \ln(\sqrt{3}+2)$$

or

$$m < \sqrt{\frac{8(\sqrt{3}+\ln(\sqrt{3}+2))}{D}} \approx 4.9\, D^{-\frac{1}{2}}.$$

Appendix A2

In order to determine the maximum of $\pi(a, b^*)$, the following results can be used:

$$\begin{aligned} \lim_{a\to 0} \frac{\partial \pi(a,b^*)}{\partial a} &= \lim_{a\to 0} \left(\frac{\frac{m}{2\sqrt{a}} e^{m\sqrt{a}} e^{\frac{m^2 D}{8}}}{(e^{m\sqrt{a}} + e^{\frac{m^2 D}{8}})^2} D - 1 \right) \to \infty \\ \lim_{a\to\infty} \frac{\partial \pi(a,b^*)}{\partial a} &= -1. \end{aligned}$$

Since $\pi(a, b^*)$ is increasing for $a = 0$ and decreasing for $a \to \infty$, and $\pi(a, b^*)$ is twice continuously differentiable, there exists at least one maximum in $a \in]0, \infty[$ that satisfies the condition $\frac{\partial \pi(a,b^*)}{\partial a} = 0$. Rearranging terms, this condition is

$$\left(e^{m\sqrt{a}}\right)^2 + \left(2 - \frac{m}{2\sqrt{a}} D\right) e^{m\sqrt{a}} e^{\frac{m^2 D}{8}} + \left(e^{\frac{m^2 D}{8}}\right)^2 = 0. \tag{9}$$

A closed form solution to (9) most likely does not exist.

Numerically the following results can be derived: For $m < 5.240602$, $\frac{\partial \pi(a,b^*)}{\partial a} = 0$ has only one solution, $a^* = b^*$. For $m = 5.240602$ it has a second and for $m > 5.240602$ it has 3 solutions, one of which is a second maximum. This, however, is only a local maximum for $m < 5.4946$; only for $m > 5.4947$ is this second maximum the global one, so that a^* is not a best response for b^*.

Appendix A3

Extending the argument of Appendix A2 to the case of n players yields the analytically derived sufficient condition for the existence of a pure strategy Nash-equilibrium

$$m < n\sqrt{\frac{2(\sqrt{3}+\ln(\sqrt{3}+2) - \ln(n-1))}{(n-1)D}}, \quad \text{for } n \leq 22.$$

The numerically derived necessary and sufficient condition is $m < \mu D^{-\frac{1}{2}}$, with μ as given in Table 1.

Table 1: Maximum value of μ for the existence of an equilibrium in pure strategies for n players

n	2	3	4	5	6	7	8	9	10	20	50	100	1000
μ	5.49	4.73	4.57	4.56	4.58	4.63	4.69	4.74	4.80	5.28	6.09	6.79	9.29

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