Economics Bulletin

Volume 33, Issue 1

A new Pareto efficient school choice mechanism

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Abstract

This paper proposes a new school choice mechanism called the recursive Boston mechanism (RBM), which is similar to the well-known Boston mechanism. While the Boston mechanism considers the reduced problem of the original problem after removing students and their assignments in the previous step, RBM considers the subproblem. We show that RBM does not satisfy strategy-proofness and stability, but satisfies Pareto efficiency. Moreover, the set of Nash equilibrium outcomes of the preference revelation game induced by RBM is equivalent to the set of stable matchings with respect to the true preferences of students.

I am grateful to one anonymous referee for a number of useful comments. I thank Koichi Suga for his guidance, Tsuyoshi Adachi for his enlightening discussion, and Tom Wansbeek for his nice help in revising this paper. All errors are my own.
Citation: Yajing Chen, (2013) "A new Pareto efficient school choice mechanism", *Economics Bulletin*, Vol. 33 No. 1 pp. 271-277.
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Submitted: July 03, 2012. **Published:** January 30, 2013.

1. Introduction

The Boston mechanism (BM) is a widely used school choice mechanism around the world. Under BM, a student's ranking of a school strongly influences his chance of being assigned to that school. Specifically, each school accepts students who rank it as their first choice and it only accepts students who rank it as their second choice when there are seats left. Loosely speaking, BM attempts to assign the maximum possible number of students to their first-choice schools and only when all such assignments have been finished does it consider assigning students to their second-choice schools. If a student is not admitted to his first-choice school, his second-choice school may be filled with students who have listed it as their first choice. Abdulkadiroğlu and Sönmez (2003)'s study, which led to renewed interest in the design and study of school choice mechanisms, pointed out that BM does not satisfies strategy-proofness and stability, which are two main desirable properties of a mechanism. To restore strategy-proofness and stability, they proposed two competing alternatives, namely the student-optimal stable mechanism (SOSM) and the top trading cycles mechanism (TTCM). Their research inspired a large amount of papers that further developed these three mechanisms. Kojima and Ünver (2012) show that BM respects preference rankings. Since respect of preference rankings is a refinement of Pareto efficiency, BM clearly satisfies Pareto efficiency.

The main purpose and contribution of this paper is to propose a new Pareto efficient school choice mechanism, called **the recursive Boston mechanism (RBM)**. In order to introduce RBM, we need the following notions. A school choice problem consists of five components: a set of students, a set of school types, a capacity vector, the preference profile of students over schools, and the priority profile of schools over students. A matching is a vector of assignments where each student can only be assigned to one school seat and one school can only be assigned to the number of students no more than its capacity. A mechanism is a function from the set of problems to the set of matchings. Given a problem, a matching, and a subset of students, a **reduced problem** is defined by removing the subset of students and their corresponding assignments under the given matching, while the preference profile remains the same. A **subproblem** with respect to the given matching and subset of students is defined by removing the subset of students and their corresponding assignments under the given matching, while the preference profile of the subproblem are defined over the set of schools with strictly positive capacity left. That is, the schools' capacities are reduced and the students' preferences are defined over the set of schools with strictly positive capacity left. While BM considers the reduced problem of the original problem after removing students and their assignments in the previous step, RBM considers the subproblem.

As was mentioned earlier, BM attempts to assign the maximum possible number of students to their first-choice schools and only when all such assignments have been finished does it consider assigning students to their second-choice schools. RBM attempts to assign the maximum possible number of students to their first-choice schools, and when all such assignments have been finished, RBM considers assigning the maximum possible number of students to their next choices that still have strictly positive capacity left. RBM turns out to share similar properties with BM. We show that the RBM does not satisfy strategy-proofness and stability, but satisfies Pareto efficiency. Moreover, the set of Nash equilibrium outcomes of the preference revelation game induced by RBM is equivalent to the set of stable matchings with respect to the true preferences of students.

The remaining of this paper is organized as follows. Second 2 introduces the basic school choice model. Section 3 presents the algorithm of the recursive Boston mechanism. Section 4 presents properties of the recursive Boston mechanism. Section 5 concludes the paper.

2. The model

Let $\mathscr{I}$ be a set of potential **students**, and $I \subset \mathscr{I}$. Let $\mathscr{C}$ be a potential set of **schools seats**, and $\mathbb{C} \subset \mathscr{C}$. For each $\mathbb{D} \subseteq \mathbb{C}$, let $C(\mathbb{D})$ be the set of **school types** and $q(\mathbb{D}) = [q_c(\mathbb{D})]_{c\in C(\mathbb{D})}$ be the **capacity vector** of all school types associated with $\mathbb{D}$, where $q_c(\mathbb{D}) \geq 1$ and $q_c(\mathbb{D})$ stands for the maximal number of students that can be assigned to it. For a student, being unmatched is denoted as being matched to the null school type $\varnothing$ with $q_\varnothing = |I|$.

Each student $i \in I$ has a single unit demand with a strict (complete, transitive, and antisymmetric) preference order P_i over $C(\mathbb{C}) \cup \{\varnothing\}$. Let $\mathscr{P}$ denote the set of all such orders. The **preference profile** of students, denoted by $P = (P_i)_{i\in I} \in \mathscr{P}^{|I|}$, is a vector of preference orders. Let $P_{I'} = (P_i)_{i\in I'}$ denote the preference profile of any subset $I' \subset I$. Let R_i denote the weak part correspondence of P_i, i.e., for all $c,d \in C(\mathbb{C}) \cup \{\varnothing\}$, cR_id implies that cP_id or $c \sim d$. For each $c \in C(\mathbb{C}) \cup \{\varnothing\}$, let $P_i(c)$ be the preference ranking of school c at P_i, i.e., if school c is the l^{th} choice of student i under P_i, then $P_i(c) = l$. Therefore, for all $c,d \in C(\mathbb{C}) \cup \{\varnothing\}$, $P_i(c) < P_i(d)$ if and only if cP_id.

Each school $c \in C(\mathbb{C})$ has a strict (complete, transitive, and antisymmetric) priority order $\succ_c$ over I, whereas $i \succ_c j$ means that student i has higher priority than student j at school c. A **priority structure** $\succ = (\succ_c)_{c\in C(\mathbb{C})}$ is a vector of priority orders.[1] Let $U_c(i) = \{j \in I | j \succ_c i\}$, i.e., $U_c(i)$ stands for the set of students who have higher priority for school c than student i.

A **matching** is a list $\mu = (\mu_i)_{i\in I}$ such that for each $i \in I$, $\mu_i \in C(\mathbb{C}) \cup \{\varnothing\}$ and $|\mu_i| = 1$, with each school c being assigned to the number of students no more than its capacity $q_c(\mathbb{C})$. For each $c \in C(\mathbb{C}) \cup \{\varnothing\}$, let $\mu_c = \{i \in I | \mu_i = c\}$ stand for the set of students assigned to school c under μ. Given a subset of students $I' \subset I$, let $\mu_{I'}$ be the set of school seats assigned to the subset I'. A school choice **problem** is denoted by $\varepsilon = (I, C(\mathbb{C}), q(\mathbb{C}), P, \succ)$. Denote the set of problems as $\mathscr{E}$ and the set of matchings as $\mathscr{M}$. A **mechanism** is a function $\phi : \mathscr{E} \to \mathscr{M}$ that maps the set of problems to the set of matchings. Denote by $\phi_i(\varepsilon)$ the school that is assigned to i by ϕ at ε. Similarly, denote by $\phi_c(\varepsilon)$ the set of students that are assigned to c by ϕ at ε.

Let P_{-i} be the preference profile of students other than i. A mechanism ϕ is **strategy-proof** if for each $\varepsilon \in \mathscr{E}$, $i \in I$ and $P'_i \in \mathscr{P}$, $\phi(I, C(\mathbb{C}), q(\mathbb{C}), P, \succ) R_i \phi(I, C(\mathbb{C}), q(\mathbb{C}), P'_i, P_{-i}, \succ)$. A mechanism ϕ is **stable** if it is non-wasteful and fair. **Non-wastefulness** means that for each $\varepsilon \in \mathscr{E}$, $i \in I$, and $c \in C(\mathbb{C}) \cup \{\varnothing\}$, $cP_i\phi_i(\varepsilon)$ implies that $|\phi_c(\varepsilon)| = q_c(\mathbb{C})$. **Fairness** means that for each $\varepsilon \in \mathscr{E}$, $i \in I$, and $c \in C(\mathbb{C})$, $cP_i\phi_i(\varepsilon)$ implies that for each $j \in \phi_c(\varepsilon)$, $j \succ_c i$. A mechanism ϕ is **Pareto efficient** if for each $\varepsilon \in \mathscr{E}$, there exist no matchings $\mu \in \mathscr{M}$ such that $\mu_i R_i \phi_i(\varepsilon)$ for all $i \in I$, and $\mu \neq \phi(\varepsilon)$.

[1] Note that a priority order can be considered as a school's "preference" over individual students. Let $P_{C(\mathbb{C})} = (P_c)_{c\in C(\mathbb{C})}$ denote the list of school "preferences" over subsets of students. We assume that for each school c, any subset of students $I' \subset I$ such that $|I'| < q_c(\mathbb{C})$, the following two conditions hold: (1) for each $i \in I \backslash I'$, $[I' \cup \{i\}] P_c I'$; and (2) for all $i,j \in I \backslash I'$, $[I' \cup \{i\}] P_c [I' \cup \{j\}]$ if and only if $i \succ_c j$. This property of priority structures is called responsiveness (see Roth 1985).

3. The recursive Boston mechanism

We need more notation to define RBM. For any subset of schools $C^{'} \subseteq C(\mathbb{C})$ and $P_i \in \mathscr{P}$, $P_i|_{C'}$ is a **projection** of P_i at $C^{'}$ if $P_i|_{C'}$ is defined as a strict order over $C^{'} \cup \{\varnothing\}$, and for all $c,d \in C^{'} \cup \{\varnothing\}$, school c has higher preference ranking than school d at $P_i|_{C'}$ if and only if school c has higher preference ranking than school d at P_i, i.e., $P_i|_{C'}(c) < P_i|_{C'}(d) \Leftrightarrow P_i(c) < P_i(d)$. For each $I^{'} \subseteq I$, we say that $P_{I'}|_{C'}$ is a projection of $P_{I'}$ at $C^{'}$ if $P_i|_{C'}$ is a projection of P_i at $C^{'}$ for all $i \in I^{'}$. Given a problem $\varepsilon = (I, C(\mathbb{C}), q(\mathbb{C}), P, \succ)$, a matching μ, and a subset of students $I \backslash I^{'}$, a reduced problem with respect to μ and $I^{'}$ is defined by removing $I \backslash I^{'}$ and their corresponding assignments, while the preference profile remains the same. Formally, a **reduced problem** of ε with respect to μ and $I^{'}$, denoted by $\varepsilon^r(I^{'}, \mu)$, is a list $\varepsilon^r(I^{'}, \mu) = (I^{'}, C(\mathbb{C} \backslash \mu_{I\backslash I'}), q(\mathbb{C} \backslash \mu_{I\backslash I'}), P_{I'}, \succ)$. Given a problem $\varepsilon = (I, C(\mathbb{C}), q(\mathbb{C}), P, \succ)$, a matching μ, and a subset of students $I \backslash I^{'}$, a subproblem of ε with respect to μ and $I^{'}$, is defined by removing $I \backslash I^{'}$ and their corresponding assignments, while the preference profile of the subproblem is defined over the remaining school types that still have strictly positive capacity left. That is, the schools' capacities are reduced and the students' preferences for the subproblem are defined over the set of schools with strictly positive capacity left. Formally, a **subproblem** of ε with respect to μ and $I^{'}$, denoted by $\varepsilon^s(I^{'}, \mu)$, is a list $\varepsilon^s(I^{'}, \mu) = (I^{'}, C(\mathbb{C} \backslash \mu_{I\backslash I'}), q(\mathbb{C} \backslash \mu_{I\backslash I'}), P_{I'}|_{C(\mathbb{C} \backslash \mu_{I\backslash I'})}, \succ)$.

Given a problem $\varepsilon \in \mathscr{E}$, **the recursive Boston mechanism**, denoted by γ, determines a matching $\gamma(\varepsilon)$ through the following algorithm:

Step 1: Only the first choices of the students are considered. For each school, consider the students who listed it as their first choice and assign seats of the school to these students one at a time following their priority order until there is no seat left or there is no student left who has listed it as his first choice. Remove students who are assigned a seat in this step and their assigned seats.

⋮

Step *k*: Consider the **subproblem** induced by the removing of students who get a seat in the previous steps and their assignments. Only the first choices of the remaining students in the **subproblem** are considered. For each school, consider the students who listed it as their first choice and assign seats of the school to these students one at a time following their priority order until there is no seat left or there is no student left who has listed it as his first choice. Remove students who are assigned a seat in this step and their assigned seats.

The algorithm terminates when all students have been assigned to a seat. In step k, if we replace "subproblem" by "**reduced problem**", then we get **the Boston mechanism**. Denote the Boston mechanism by β.

EXAMPLE 1 The problem ε is defined as follows. Let $I = \{i_1, i_2, i_3, i_4\}$, $\mathbb{C} = \{c_1, c_2, c_3\}$, and the capacity of each school is one. The preferences of students and the priority orders of schools are listed below:

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	$\succeq_{c_1}$	$\succeq_{c_2}$	$\succeq_{c_3}$
c_1	c_1	c_3	c_3	i_1	i_2	i_1
c_2	c_3	c_2	c_1	i_2	i_3	i_2
c_3	c_2	c_1	c_2	i_3	i_1	i_4
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	i_4	i_4	i_3

The algorithm of the recursive Boston mechanism, denoted by γ, results in the following matching:

$$\gamma(\varepsilon) = \left(\begin{array}{cccc} i_1 & i_2 & i_3 & i_4 \\ c_1 & c_2 & \varnothing & c_3 \end{array} \right)$$

The algorithm of the Boston mechanism, denoted by β, results in the following matching:

$$\beta(\varepsilon) = \left(\begin{array}{cccc} i_1 & i_2 & i_3 & i_4 \\ c_1 & \varnothing & c_2 & c_3 \end{array} \right)$$

Note that BM and RBM both assign c_1 to i_1 and c_3 to i_4 in the first step. The difference occurs in step 2. In step 2, BM considers the second choices of i_2 and i_3, and assigns c_2 to i_3 because i_2 puts c_2 in a lower preference ranking than i_3 does. On the contrary, RBM considers the subproblem by removing c_1, c_3, i_1 and i_4. In the updated preference profile, i_2 and i_3 both list c_2 as their first choice. Because i_2 has higher priority than i_3 for c_2, c_2 is assigned to i_2 under RBM.

4. Properties of the recursive Boston mechanism

Proposition 1. *γ is not stable.*

Proof. We see from example 1 that $\gamma(\varepsilon)$ is not stable because $c_3 P_{i_2} c_2$, $i_4 \in \gamma_{c_3}(\varepsilon)$, but $i_2 \succ_{c_3} i_4$. □

Proposition 2. *γ is not strategy-proof.*

Proof. In example 1, if student i_2 reports the preference $P^{'}_{i_2} : c_3 P^{'}_{i_2} \varnothing$, then he will be assigned to school c_3 and be better off. Thus, γ is not strategy-proof. □

Proposition 3. *γ is Pareto efficient.*

Proof. Consider the procedure of RBM. Any student who leaves at Step 1 is assigned his top choice and cannot be made better off. Any student who leaves at Step k is assigned his top choice among those schools remaining at Step k ($k \geq 2$). Since preferences are strict, he cannot be made better off without hurting someone who left at Step 1 to $k-1$. By the same logic, no student can be made better off without hurting someone who left at the earlier steps. Therefore, γ is Pareto efficient. □

Note that since γ is not strategy-proof, it is Pareto efficient with respect to the reported preference profile. Ergin and Sönmez (2006) prove that for BM, the set of Nash equilibrium outcomes of the preference revelation game induced by BM is equal to the set of stable matchings under the true preferences of students. One can easily imagine that RBM also satisfies this property and proofs for the next proposition are almost the same with proofs for theorem 1 in Ergin and Sönmez (2006).

Proposition 4. *Let $\widehat{P}$ be the list of true student preferences and consider the preference revelation game induced by RBM. The set of Nash equilibrium outcomes of this game is equal to the set of stable matchings under the true preferences $\widehat{P}$.*

Proof. Let P be an arbitrary strategy profile and let μ be the resulting outcome of RBM. Suppose that μ is not stable under the true preferences $\widehat{P}$. Then, we conclude that either there exist $i \in I$ and $c \in C(\mathbb{C}) \cup \{\varnothing\}$ with $c\widehat{P}_i\mu_i$ such that $|\mu_c| < q_c(\mathbb{C})$, or there exist $i, j \in I$ and $c \in C(\mathbb{C})$ with $c\widehat{P}_i\mu_i$ and $j \in \mu_c$ such that $i \succ_c j$. By the procedure of γ, $P_i(c) \neq 1$ for otherwise he would be assigned to a seat at c. Let $P_i^{'}$ be any strategy where $P_i^{'}(c) = 1$. It is easy to see that $\gamma_i(I, C(\mathbb{C}), q(\mathbb{C}), P_i^{'}, P_{-i}, \succ) = c$. Therefore, P is not a Nash equilibrium strategy profile and μ is not a Nash equilibrium outcome. Hence, any Nash equilibrium outcome should be stable under the true preferences $\widehat{P}$.

Conversely, let μ be an arbitrary stable matching under the true preferences $\widehat{P}$. Consider a preference profile P where for each $i \in I$, $P_i(\mu_i) = 1$. We then prove that P is a Nash equilibrium strategy profile. Consider a $c \in C(\mathbb{C})$ such that $c \neq \mu_i$ and $cP_i\mu_i$. Since μ is stable and $P_i(\mu_i) = 1$, we have that $|\mu_c| = q_c(\mathbb{C})$, and for each $j \in \mu_c$, $P_j(c) = 1$ and $j \succ_c i$. Therefore, given P_{-i}, there is no way student i is assigned a seat in school c even if he ranks c first. Moreover, it is easy to see that $\gamma(I, C(\mathbb{C}), q(\mathbb{C}), P, \succ) = \mu$. Therefore, P is a Nash equilibrium strategy profile and μ is the corresponding Nash equilibrium outcome. Since μ is an arbitrarily chosen stable matching under the true preferences $\widehat{P}$, any stable matching under the true preferences $\widehat{P}$ is a Nash equilibrium outcome. □

5. Discussion and concluding remarks

The seminal paper of Abdulkadiroğlu and Sönmez (2003) introduced three well-known school choice mechanisms, i.e., the SOSM, TTCM and BM. The current paper proposes the recursive Boston mechanism (RBM) which is similar to BM. We prove that RBM shares many similar properties with BM. RBM is neither stable nor strategy-proof, but is Pareto efficient. Moreover, The set of Nash equilibrium outcomes of the preference revelation game induced by RBM coincides with the set of stable matchings under the true preferences. Future work is needed to further investigate the theoretical and empirical performances of RBM, especially the difference between RBM and the other school choice mechanisms.

When each school has its own strict priority over students, BM was criticized for its lack of strategy-proofness and stability and has been replaced by SOSM in many places. Miralles (2008) proposes a variation of BM when schools do not have strict priorities. He proves that when every school shares the same priority, BM outperforms SOSM according to several ex ante efficiency criteria. The current paper restricts the priorities to be strict. Future work is needed to extend RBM to the environment when schools have no strict priorities.

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