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Flexible capital-labor assignment model

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Abstract

This paper constructs an assignment model that allocates a variable number of labor to a variable amount of capital. Sattinger introduced an assignment model in which a worker is coupled with a machine. In Sattinger's model, workers and machines differ in quality, e.g., ability, productivity, efficiency, or size. In the model in the present paper, the number of workers and the number of machines also differ from assignment to assignment. First, I introduce a model that assigns a different number of workers with different abilities to a different amount of capital with the same quality, then extend that model to a case in which quality is different among the capital.

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1. Introduction

An assignment model is in a class of models that assumes there are heterogeneous resources and considers the assignment of these resources. The resources differ in quality, e.g., productive ability, size, efficiency, preference. The difference in quality between resources causes assignment problems. How a certain kind of resource is assigned to another kind of resource determines productivity and output level. As a result, when there are heterogeneous resources, assignments become a crucial element that determines productivity.

There are many kinds of assignment models. In Roy (1951), Roy considered models in which there were two or more kinds of jobs and in which agents differed in productive ability in those jobs. In that model's economy, each agent has to choose the job that would provide the highest wages. The method of assigning each agent to each job determined the total output and productivity in the economy.

In Sattinger (1979), Sattinger developed a one to one matching model. A worker is assigned to a machine and produces goods. Workers differ in productive ability and machines differ in size. Productive ability and size affect the amount of output. In this economy, Sattinger developed a method for deriving a wage function and a rental cost function.

The model described in Sattinger (1979) has a number of extensions and applications, including the ones used in this paper. Teulings examined the effect of difference in worker's ability and job complexity (see Teulings (1995, 2005). Costrell and Loury extended the Sattinger model to the cace in which jobs have a hierarchical structure (see Costrell (2004)). Tervio applied Sattinger's model to the wages of CEOs (see Tervio (2008)).

In this paper, I construct models in which a variable number of workers are assigned to a variable amount of capital. Workers differ in productive ability and capital differs in quality. In this case, both quantity and quality affect the assignment, and therefore, the output. Therefore, this is more general assignment problem.

2. Assumption

Let $x \in [0,1]$ denote a worker's ability and $y \in [0,1]$ denotes quality of capital. And let $g(x)$ be the density function of workers with respect to ability x and $k(y)$ be the density function of capital with respect to quality y. The density functions are atomless with full support and twice differentiable.

The total number of labor and the total amount of capital are both one, i.e., $\int_0^1 g(x)dx = 1$ and $\int_0^1 k(y)dy = 1$.

Let $q(x)$ be the amount of capital to which a worker with ability x is assigned and $Q(x)$ be the amount of capital the workers with ability between 0 and x use. Thus, $Q(x) = \int_0^x q(z)g(z)dz$ and $Q'(x) = q(x)g(x)$.

The production functions takes the form $f(x,q)$, in a homogeneous capital case, or $f(x,y,q)$, in a heterogeneous capital case. Properties of the production functions are $f_x > 0$, $f_y > 0$, $f_q > 0$ and $0 < \frac{f_q}{f}q < 1$. The total output in this economy is $\int_0^1 f(x,q(x))g(x)dx$ and $\int_0^1 f(x,y(x),q(x,y))$.

3. Models

3.1 Homogeneous capital case

Assume that the production function is a Cobb-Douglas form, i.e., $f(x,q(x)) = x^\alpha q(x)^\gamma$. Noting $q(x) = \frac{Q'(x)}{g(x)}$, the output maximization problem is

$$\begin{aligned} \textbf{Max} \quad & \int_0^1 x^\alpha Q'(x)^\gamma g(x)^{(1-\gamma)}dx. \qquad (1) \\ s.t. \quad & \int_0^1 g(x)dx = 1. \end{aligned}$$

Solving (1), the optimal assignment is

$$\begin{aligned} Q(x) &= \frac{H(x)}{H(1)}, \\ where \;\; H(x) &= \int_0^x h(z)dz = \int_0^x z^{\frac{\alpha}{1-\gamma}} g(z)dz. \end{aligned}$$

The optimal output of the model, the wage function, $w(x)$, and the rental cost, r, are

$$\begin{aligned} \int_0^1 x^\alpha Q'(x)^\gamma g(x)^{(1-\gamma)}dx &= H(1)^{(1-\gamma)}, \\ w(x) &= \frac{(1-\gamma)x^{\frac{\alpha}{1-\gamma}}}{H(1)^\gamma}, \qquad (2) \\ r &= \gamma H(1)^{(1-\gamma)}. \qquad (3) \end{aligned}$$

3.2 Heterogeneous capital case

Assume that the production function is a Cobb-Douglas form, i.e., $f(x, y, q(x, y)) = x^{\alpha}y(x)^{\beta}q(x, y)^{\gamma}$. The amount of capital the workers with ability from 0 to $\overline{x}$ use and the amount of capital with quality between 0 to $y(\overline{x})$ are equel, i.e., $\int_0^{\overline{x}} q(x, y(x))g(x)dx = \int_0^{y(\overline{x})} k(y)dy$. Differentiating, we get $q(x, y) = \frac{y'(x)k(y)}{g(x)}$. The output maximization problem is

$$\begin{aligned} \textbf{Max} \quad & \int_0^1 x^{\alpha}y(x)^{\beta}y'(x)^{\gamma}k(y)^{\gamma}g(x)^{(1-\gamma)}dx, \\ s.t. \quad & \int_0^1 g(x)dx = 1, \int_0^1 k(y)dy = 1. \end{aligned} \tag{4}$$

The optimal assignment to (4) is

$$\begin{aligned} \frac{\int_0^{y(\overline{x})} z^{\frac{\beta}{\gamma}}k(z)dz}{\int_0^1 z^{\frac{\beta}{\gamma}}k(z)dz} &= \frac{\int_0^{\overline{x}} z^{\frac{\alpha}{1-\gamma}}g(z)dz}{\int_0^1 z^{\frac{\alpha}{1-\gamma}}g(z)dz}. \\ y'(x) &= y(x)^{-\frac{\beta}{\gamma}}k(y)^{-1}x^{\frac{\alpha}{1-\gamma}}g(x)H_v(1), \\ where \;\; H_v(1) &= \frac{H_y(1)}{H_x(1)} = \frac{\int_0^1 z^{\frac{\beta}{\gamma}}k(z)dz}{\int_0^1 z^{\frac{\alpha}{1-\gamma}}g(z)dz}. \end{aligned}$$

The optimal output, the wage function and the rental cost function are

$$\begin{aligned} & \int_0^1 x^{\alpha}y(x)^{\beta}y'(x)^{\gamma}k(y)^{\gamma}g(x)^{(1-\gamma)}dx \\ & = \int_0^1 x^{\frac{\alpha}{1-\gamma}}g(x)H_v(1)^{\gamma}dx = H_x(1)H_v(1)^{\gamma}, \\ w(x) &= (1-\gamma)x^{\frac{\alpha}{1-\gamma}}H_v(1)^{\gamma}, \qquad (5) \\ r(y) &= \gamma H_v(1)^{\gamma-1}y(x)^{\frac{\beta}{\gamma}}. \qquad (6) \end{aligned}$$

3.3 Generalized case

In previous models, the production functions are a Cobb-Douglas form. We relax the assumption here. Let $\theta(q)$ be the contribution of capital per worker and $\eta(\rho) = \frac{\theta'(q)}{\theta(q)}q(x)$ be the elasticity of output with respect to an amount of capital per worker.[1] Assume that the output a worker with ability x produces is $f(x,y)\theta(q)$. The output maximization problem is

$$\begin{aligned} \textbf{Max} \quad & \int_0^1 f(x,y)\theta\left(\frac{k(y)y'(x)}{g(x)}\right) g(x)dx, \qquad (7) \\ s.t. \quad & \int_0^1 g(x)dx = 1, \int_0^1 k(y)dy = 1. \end{aligned}$$

Solving (7), the wage function and the rental cost function are (see **??**)

$$w(x) = (1-\eta(\rho))\, f(x,y)\theta(\rho), \qquad (8)$$

$$r(y) = f(x,y)\theta'(\rho). \qquad (9)$$

From (2), (3), (5), (6), (8), and (9), we get the next propositon.

Proposition 1. *Under the optimal assignment, the distribution between labor and capital at any point on the optimal pass depends on the value of $\eta(q)$. The proportion of distribution between labor and capital is $1-\eta$:η.*

3.4 Discussion

In these kinds of models, it is necessary that a model must satisfy two no arbitrage conditions. One is a no arbitrage condition regarding optimal assignment. The assignment must produce outputs at least as large as the other assignments. The other is a no arbitrage condition regarding optimal prices. In order for the optimal assignment to be stable, suppose that the optimal assignment is written as $y(x)$ and $x(y)$; the optimal assignment must satisfy

$$\begin{aligned} & f(x(y),y) - w(x(y)) \geq f(x',y) - w(x'), \forall x' \in x, \\ & f(x,y(x)) - r(y(x))q(x,y(x)) \geq f(x,y') - r(y')q(x,y'), \forall y' \in y. \end{aligned}$$

There is one difference between Sattinger's models and the models in this paper. In Sattinger's models, the optimal assignment is at least as good as the others. However in the model's in the present paper, the marginal productivities of capital with some quality are constant regardless of where these capital are assigned. As a result, the models presented in this paper achieve more productive assignment unless one to one capital labor matching is optimal.

[1] The elasticities of output with respect to an amount of capital per worker and the contribution of capital per worker are the same.

4. Conclusion

In this paper, I constructed a capital labor assignment in which that capital is assigned to labor on a flexible basis. This is a natural extention of Sattinger (1979). More flexible assignments can possibly increase the total output. Flexible assignments make wages and rental costs dependent on the contributions of resources and the elasticity of output with respect to an amount of capital per worker or an number of worker per capital.

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