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Remarks on Young's theorem

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Abstract

In this paper we analyze the simple case of voting over two alternatives with variable electorate. Our findings are (a) the axiom of continuity is redundant in the axiomatization of scoring rules in Young (1975), SIAM J. Appl. Math. 28: 824-838, (b) the smaller set of axioms characterize this voting rule when indifferences are allowed in the voters' preferences, (c) a version of May's theorem can be derived from this last result, and finally, (d) in each of these results, axioms of neutrality and cancellation property can be used interchangeably.

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1. Introduction

In this paper we reconsider the problem of axiomatizing scoring rules. Early results on this problem are due to Smith (1973) and Young (1975). They characterized social welfare and social choice functions, respectively, as scoring rules with four basic axioms: anonymity, neutrality, consistency (or separability, or reinforcement) and continuity (or Archimedian, or overwhelming majority). Following them, Myerson (1995) showed that essentially the same set of axioms characterize scoring rules even if some of the assumptions of Smith (1973) and Young (1975) are weakened.

Our objective in this paper is to point out an important detail that has seemingly been ignored in this literature: in the special case of two alternatives the continuity axiom in Young (1975) is redundant in the axiomatization of scoring rules. Hence, our main result is (Theorem 2 in Sect. 3.1) "When there are two alternatives, a social choice function is anonymous, neutral and consistent if and only if it is a simple scoring function." We also show that the same result holds, i.e. the smaller set of axioms characterize this voting rule when indifferences are allowed in the voters' preferences (Theorem 3 in Sect. 3.2). Moreover, from this result we derive another result (Theorem 4 in Sect. 3.2) that can be seen as a variant of May's theorem in May (1952), and hence establish a formal connection between the two classic results, Young's Theorem and May's Theorem. Finally, we also show that in each of our results, axioms of neutrality and cancellation property can be used interchangeably (Proposition 1 and 2).

In the next section we introduce our notation and the main definitions. Section 3 gives the main results and the last section concludes.

2. The preliminaries

Let $\mathbb{R}^n$ denote the set of all n-tuples of real numbers and let $\mathbb{R}^n_+$ be its nonnegative orthant. The notions of weak (and associated indifference relation) and strict preferences over a set B are defined as usual and when $a \in B$ is weakly, strictly preferred and indifferent to $b \in B$, we write $a \gtrsim b, a \succ b$ and $a \sim b$, respectively. A transposition on set $B = \{a, b\}$, that is a permutation that exchanges the roles of a and b, is denoted as $a \leftrightarrows b$.

Our main setting follows closely that of Young (1975). Let $A = \{a_1, ..., a_m\}$ be the set of alternatives. Let $\mathbb{N}$ denote the set of nonnegative integers which constitute names for the voters and let $\mathbb{P}$ be the set of all preference orders (strict) on A. For any finite $V \subset \mathbb{N}$, a *profile* is a function from V to $\mathbb{P}$ and a *social choice function* (SCF) is a function from set X of all profiles to the family of non-empty subsets of $A, 2^A \backslash \{\emptyset\}$. A SCF is said to be *anonymous* if it depends only on the number of voters associated with each preference order. We can represent the domain of an anonymous SCF by $\mathbb{N}^{m!}$, i.e. the set of all $m!$-tuples with nonnegative integer coordinates, indexed by $\mathbb{P}$, where for any $x \in \mathbb{N}^{m!}$ and any $p \in \mathbb{P}, x_p$ represents the number of voters having preference order p. Let S_m be the group of permutations of the index set $\{1, 2, ..., m\}$. Each $\sigma \in S_m$ induces permutations of the alternatives (which we also denote by σ), and hence profiles, in the natural way. We say that SCF is *neutral* if $f \circ \sigma = \sigma \circ f$ for all $\sigma \in S_m$. An anonymous SCF f is *consistent* if $\forall x', x'' \in \mathbb{N}^{m!}$ such that $f(x') \cap f(x'') \neq \emptyset$, $f(x' + x'') = f(x') \cap f(x'')$, and it is *continuous* if whenever $f(x) = \{a_i\}$, $\forall y \in \mathbb{N}^{m!}$ there is a sufficiently large integer n such that $f(y + n'x) = \{a_i\}$ for $n' \geq n$.

We say that a SCF f has the *cancellation property* if whenever x is a profile such that the number of voters preferring a_i to a_j equals that of preferring a_j to a_i for all pairs

$a_i \neq a_j$, then $f(x) = A$. A SCF is a *simple scoring function*, denoted by f^s, if there is a vector $s = (s_1, ..., s_m) \in \mathbb{R}^m$ of scores such that for any given profile, it assigns a score of s_i to each voter's *i'th* most preferred alternative and chooses the alternative(s) with the highest total score at that profile.[1] A SCF is *trivial* (f^*) if $\forall x \in \mathbb{N}^{m!}, f^*(x) = A$. Note that f^* is a simple scoring function with $s = (0, ..., 0)$.

The following result is known as Young's Theorem (Theorem 1.(ii) in Young, 1975):

Theorem 1 *Let f be a SCF. Then, f is anonymous, neutral, consistent and continuous if and only if it is a simple scoring function.*

3. The axioms for scoring rules when $m = 2$

3.1. Two remarks on Young's theorem

Before we consider the case of $m = 2$, we prove the following Lemma which holds for any finite m.

Lemma 1 *Suppose f is an anonymous, neutral and consistent SCF. Then,*

(a) *f is either trivial or it contains all the singletons of $2^A \backslash \{\emptyset\}$ in its range:* $\forall a_i \in A$, $\exists x \in \mathbb{N}^{m!}$ *such that* $f(x) = \{a_i\}$.

(b) *Let $e = (1, ..., 1) \in \mathbb{N}^{m!}$. Then, $\forall n \in \mathbb{N}$, $f(ne) = A$.*

Proof. **(a)** Note that when $m = 1$ the result is trivial. Suppose $m \geq 2$. We show that $R(f)$, the range of f, includes at least one singleton $\{a_i\}$. Then, the result follows by neutrality. When $m = 2$, the claim is trivial since $f \neq f^*$ immediately implies that $\exists x \in \mathbb{N}^{2!}$ such that $f(x) = \{a_i\}$ for some $a_i \in \{a_1, a_2\}$. Let $m \geq 3$ and suppose $R(f)$ does not contain any singleton. Then, we claim that it can't have any 2-element sets, 3-element sets,..., $(m-1)$-element sets. Because if $R(f)$ has a 2-element set $\{a_i, a_j\}$, then by neutrality it has another 2-element set $\{a_i, a_k\}$. Then for $x' \in f^{-1}(\{a_i, a_j\})$ and $x'' \in f^{-1}(\{a_i, a_k\})$, consistency implies that $f(x' + x'') = \{a_i\}$, which is a singleton. Hence, we reach to a contradiction. Similarly, we conclude that $R(f)$ can't have any k-element sets, for $3 \leq k < m$. But then, $f = f^*$.

(b) We show that $f(e) = A$. Then the result follows by consistency. Note that $e \in \mathbb{N}^{m!}$ is invariant under all permutations in S_m. Then $f(e)$ must be so by neutrality. But in $2^A \backslash \{\emptyset\}$, the only set with that property is A. ■

We remark here that Lemma 1 **(b)** is already established in Young (1975). Let us now consider the case of $m = 2$. Our main result is the following:

Theorem 2 *Let $m = 2$ and let f be a SCF. Then, f is anonymous, neutral and consistent if and only if it is a simple scoring function.*

Proof. Since the IF part is easy to verify we prove the ONLY IF part. For any $x = (x_1, x_2) \in \mathbb{N}^2 \subset \mathbb{R}^2_+$, let x_1 be the number of voters with preference $p_1 : a_1 \succ a_2$ and let x_2 be that with $p_2 : a_2 \succ a_1$. We shall partition $\mathbb{N}^2 \subset \mathbb{R}^2_+$ as follows:

$$\begin{aligned} D &= \{x \in \mathbb{N}^2 : x_1 = x_2\}; \\ D' &= \{x \in \mathbb{N}^2 : x_1 > x_2\}; \\ D'' &= \{x \in \mathbb{N}^2 : x_1 < x_2\}. \end{aligned}$$

[1] Note that scoring rules constitute rather general class of voting procedures. In particular, the possibility of assigning lower scores to more preferred alternatives is allowed.

Then by Lemma 1 **(b)**, $\forall x \in D, f(x) = \{a_1, a_2\}$. We claim that $\forall x' \in D', f(x') = f(1,0)$. Suppose $\exists x' \in D'$ such that $f(x') = f(1,0)$. Then, by consistency

$$f(x' + (1,0)) = f(x'_1 + 1, x'_2) = f(1,0)$$

and

$$f(x' + (1,1)) = f(x'_1 + 1, x'_2 + 1) = f(1,0).$$

Hence, for any such $x' \in D'$, its two immediate neighbors, one on the right side and one on the upper right side, take the same value. Since $(1,0) \in D'$, this proves our claim. By neutrality, then $\forall x'' \in D'', f(x'') = f(0,1)$. Then by Lemma 1 **(a)**, f is either f^*, or f_1:

$$f_1(x) = \begin{cases} A \text{ if } x \in D \\ \{a_1\} \text{ if } x \in D' \\ \{a_2\} \text{ if } x \in D'' \end{cases},$$

or f_2:

$$f_2(x) = \begin{cases} A \text{ if } x \in D \\ \{a_2\} \text{ if } x \in D' \\ \{a_1\} \text{ if } x \in D'' \end{cases}.$$

Since f_1 and f_2 correspond to f^s with $s_1 > s_2$ and $s_1 < s_2$ respectively, the proof is completed. ∎

Let us show that one can use the axioms of neutrality and cancellation property interchangeably in Theorem 2.

Proposition 1 *Let $m = 2$ and let f be an anonymous and consistent SCF. Then, f has cancellation property if and only if it is neutral.*

Proof. IF: We have shown in Lemma 1 **(b)** that anonymity, consistency and neutrality imply that, $\forall x \in D, f(x) = \{a_1, a_2\}$. Since $m = 2$, any profile $x \in \mathbb{N}^2$ is such that the number of voters who prefers $a_1 \succ a_2$ is same as that of who prefers $a_2 \succ a_1$, if and only if $x \in D$. Hence, f satisfies the cancellation property.

ONLY IF: Suppose f satisfies anonymity, consistency and cancellation property, but not neutrality: $\exists y \in \mathbb{N}^2$ such that $f \circ \sigma'(y) \neq \sigma' \circ f(y)$ for some $\sigma' \in S_2$. The only candidate for such $\sigma' \in S_2$ is $\sigma' : a_1 \leftrightarrows a_2$. Note that if one of $f \circ \sigma'(y) \in 2^A \backslash \{\emptyset\}$ and $f(y) \in 2^A \backslash \{\emptyset\}$ is A, then consistency implies that they both must be A, since $\sigma'(y) + y \in D$ and by cancellation property, $f(\sigma'(y) + y) = A$. But then $f \circ \sigma'(y) = \sigma' \circ f(y) = A$ which contradicts to our assumption. Hence, none of $f \circ \sigma'(y) \in 2^A \backslash \{\emptyset\}$ and $f(y) \in 2^A \backslash \{\emptyset\}$ is A. Suppose $f \circ \sigma'(y) = \{a_i\}$ and $f(y) = \{a_j\}$ for $i, j \in \{1, 2\}, i \neq j$. Then $f \circ \sigma'(y) = \sigma' \circ f(y) = \{a_i\}$, which is a contradiction. Hence, the only possibility left is $f \circ \sigma'(y) = f(y) = \{a_i\}$ for some $i \in \{1, 2\}$. Then by consistency $f(\sigma'(y) + y) = \{a_i\}$, which contradicts to cancellation property. This completes our proof. ∎

In our opinion, redundancy of the continuity axiom in Theorem 1 when $m = 2$ is not so obvious until one proves Theorem 2. However, one can also verify it directly from Theorem 1.(i) in Young (1975) which states that a SCF is anonymous, neutral and consistent if and only if it is a (*composite*) scoring function. Provided that Theorem 1.(i) is proven, it suffices to notice that when $m = 2$, a *composition* $g = f^{s^2} \circ f^{s^1}$ of two simple scoring functions f^{s^1}, f^{s^2}, defined as $g(x) = f^{s^1}(x)$ if $f^{s^1}(x) \subseteq A$ is a singleton set, otherwise apply f^{s^2} to break the ties in $f^{s^1}(x)$, is a simple scoring function. Note that there are two possibilities: either f^{s^1} is trivial or it is not. Suppose $f^{s^1} = f^*$. Then, since $g = f^{s^2} \circ f^* = f^{s^2}$, g is a simple scoring function. Now suppose $f^{s^1} \neq f^*$. Then, since when $m = 2, f^{s^1}(x)$ produces ties if and only if $x \in D$ and since $f^{s^2}(x) = A$ for $x \in D$, we conclude that $g = f^{s^1}$, hence g is a simple scoring function.

3.2. Allowing for indifferences in voters' preferences

Let us now change our initial setting by allowing indifferences in the individual's preferences, hence enlarging the domain of SCF. Sets A, S_m and $\mathbb{N}$ are defined as above in Sect. 2. Let $\mathbb{W}$ be the set off all weak preference orders on A. For any finite $V \subset \mathbb{N}$, an *extended profile* is a function from V to $\mathbb{W}$ and an *extended social choice function* (ESCF) is a function from set Y of all extended profiles to the set $2^A \backslash \{\emptyset\}$. An ESCF is said to be *anonymous* if it depends only on the number of voters associated with each preference order. We can represent the domain of an anonymous ESCF by $\mathbb{N}^{\#\mathbb{W}}$. The notions of neutrality, consistency and cancellation property for an ESCF are defined analogously to that for a SCF.

Since scoring rules are initially defined for profiles with strict preferences in Sect. 2, it needs to be generalized. As we are eventually interested in the case of $m = 2$, we impose a rather weak condition in our generalization: whenever alternatives are indifferent to each others at a given preference they must receive the same score (for a more specific generalization which applies to the case of any finite m, see Vorsatz, 2008). So when $m = 2$, an ESCF is a *simple scoring function*, denoted by F^s, if there is a vector $s = (s_1, s_2, s_3) \in \mathbb{R}^3$ of scores such that for any given profile, it assigns a score of s_i to each voter's $i'th$ strictly most preferred alternative, for $i = 1, 2$, and assigns a score of s_3 to each voter's indifferent alternatives, and chooses the alternative(s) with the highest total score at that profile. An ESCF is *trivial (F^*)* if $\forall x \in \mathbb{N}^{\#\mathbb{W}}, F^*(x) = A$. Note that when $m = 2$, F^* is a simple scoring function with $s = (0, 0, 0)$.

Theorem 3 *Let $m = 2$ and let F be an ESCF. Then, F is anonymous, neutral and consistent if and only if it is a simple scoring function.*

Proof. IF: F^s is clearly anonymous since the outcome of F^s depends only on the total scores and that in turn depends only on the number of voters associated with each preference. F^s is neutral since exchanging the roles of a_1 and a_2 is same as exchanging the total scores received by each. F^s is consistent since the total score received by a_i under $x + y \in \mathbb{N}^3$ is the sum of the scores received under each of $x, y \in \mathbb{N}^3$, for any $x, y \in \mathbb{N}^3$ and $i = 1, 2$.

ONLY IF: For any $x = (x_1, x_2, x_3) \in \mathbb{N}^3 \subset \mathbb{R}^3_+$, let x_1, x_2 and x_3 be the number of voters with the preferences $p_1 : a_1 \succ a_2, p_2 : a_2 \succ a_1$ and $p_3 : a_1 \sim a_2$, respectively. We shall partition $\mathbb{N}^3 \subset \mathbb{R}^3_+$ as follows:

$$
\begin{aligned}
D_3 &= \{x \in \mathbb{N}^3 : x_1 = x_2, x_3 \in \mathbb{N}\};\\
D'_3 &= \{x \in \mathbb{N}^3 : x_1 > x_2, x_3 \in \mathbb{N}\};\\
D''_3 &= \{x \in \mathbb{N}^3 : x_1 < x_2, x_3 \in \mathbb{N}\}.
\end{aligned}
$$

Firstly, note that $\forall x \in D_3, F(x) = \{a_1, a_2\}$ since $x \in D_3$ is invariant under all permutations in S_2 (recall that the indifference relation is symmetric) and by neutrality so must be $F(x)$. But the only set with that property in $\{a_1\}, \{a_2\}$ and $\{a_1, a_2\}$ is the very last one.

Let $P_n \subset \mathbb{N}^3$ be defined as follows: $\forall n \in \mathbb{N}, P_n = \{x \in \mathbb{N}^3 : x_1, x_2 \in \mathbb{N}, x_3 = n\}$. We claim that for $n \geq 1$, if $F(x) = F(1,0,0)$, $\forall x \in P_{n-1} \cap D_3'$, then $F(x) = F(1,0,0)$, $\forall x \in P_n \cap D_3'$. Suppose $y \in P_{n-1} \cap D_3'$ and $F(y) = F(1,0,0)$. Then by consistency, $F(y + (0,0,1)) = F(y) = F(1,0,0)$ since $(0,0,1) \in D_3$ and $F(0,0,1) = A$. But for $n \geq 1, P_n = \{x \in \mathbb{N}^3 : x = y + (0,0,1), y \in P_{n-1}\}$ and this proves our claim.

Recall that we already showed in the proof of Theorem 2 that $F(x) = F(1,0,0)$, $\forall x \in P_0 \cap D_3'$. Hence, we conclude that $\forall x \in D_3', F(x) = F(1,0,0)$, and then by neutrality, $\forall x \in D_3'', F(x) = F(0,1,0)$. To complete the proof, suppose $F \neq F^*$, then F includes all singletons in its range since $F \neq F^*$ implies that $\exists x \in \mathbb{N}^3$ such that $F(x) = \{a_i\}$ for some $a_i \in A$ and hence by neutrality, $\exists x^i \in \mathbb{N}^3$ such that $F(x^i) = \{a_i\}$, for $i = 1, 2$. Combining our last observation with the above conclusions, we have established that F is either F^*, or F_1 :

$$F_1(x) = \begin{cases} A \text{ if } x \in D_3 \\ \{a_1\} \text{ if } x \in D_3' \\ \{a_2\} \text{ if } x \in D_3'' \end{cases},$$

or F_2 :

$$F_2(x) = \begin{cases} A \text{ if } x \in D_3 \\ \{a_2\} \text{ if } x \in D_3' \\ \{a_1\} \text{ if } x \in D_3'' \end{cases}.$$

Since F_1 and F_2 correspond to F^s with $s_1 > s_2$ and $s_1 < s_2$ respectively, the proof is completed. ■

Let us now derive a variant of May's theorem from Theorem 3 above. First, we need to introduce some more properties for anonymous ESCFs. For any extended profile $x \in \mathbb{N}^3$, let $N(a_i, x) \in \mathbb{N}$ be the number of voters who prefers (weakly) a_i to a_j at $x \in \mathbb{N}^3$, for $i, j \in \{1, 2\}$ and $i \neq j$. An anonymous ESCF is a *simple majority rule (F^M)* if

$$F^M(x) = \begin{cases} \{a_i\} \text{ if } N(a_i, x) > N(a_j, x) \\ \{a_1, a_2\} \text{ if } N(a_i, x) = N(a_j, x) \end{cases}$$

for $i, j \in \{1, 2\}$, $i \neq j$. It is an *inverse simple majority rule (F^{-M})* if

$$F^{-M}(x) = \begin{cases} \{a_i\} \text{ if } N(a_i, x) < N(a_j, x) \\ \{a_1, a_2\} \text{ if } N(a_i, x) = N(a_j, x) \end{cases}$$

for $i, j \in \{1, 2\}$, $i \neq j$. Finally, anonymous ESCF is *positive responsive to voter addition* (positive responsiveness) if whenever $a_i \in F(x)$ for $x \in \mathbb{N}^3$, and $y \in \mathbb{N}^3$ is obtained from $x \in \mathbb{N}^3$ by adding one more voter with preferences of $a_i \succ a_j$, we have $F(y) = \{a_i\}$, for $i, j \in \{1, 2\}$, $i \neq j$. The second part of the following result is a variant of May's Theorem (May (1952)):

Theorem 4 *Let $m = 2$ and let F be an ESCF. Then,*

(a) *F is anonymous, neutral and consistent if and only if it is either trivial, or a simple majority rule, or an inverse majority rule and*

(b) *F is anonymous, neutral and positive responsive if and only if it is a simple majority rule.*

Proof. Since the IF parts are easy we prove the ONLY IF parts.

(a) By definition, F_1 and F_2 in the proof of Theorem 3 correspond to simple majority rule and inverse simple majority rule, respectively.

(b) We present two proofs.

1. Let us show that positive responsiveness with anonymity and neutrality imply consistency. From the proof of Theorem 3, we know that anonymity and neutrality imply that, $\forall x \in D_3, F(x) = \{a_1, a_2\}$. For any $x = (x_1, x_2, x_3) \in \mathbb{N}^3$, let $x_{D_3} \in D_3$ be defined as $x_{D_3} = (\min\{x_1, x_2\}, \min\{x_1, x_2\}, x_3)$. We can write x as $x = x_{D_3} + (x - x_{D_3})$. We claim that $F(x) = F(x - x_{D_3})$. Note that if $x \in D_3$, then $F(x) = F(0) = A$, hence the claim is true. Suppose $x \notin D_3$. Then one can generate $x \in \mathbb{N}^3$ from $x_{D_3} \in D_3$ and $x - x_{D_3} \in \mathbb{N}^3$ from $0 \in D_3$ by one and the same procedure: if $x_1 > x_2$, adding $|x_1 - x_2| > 0$ voters, one at a time, who strictly prefers a_1 to a_2, if otherwise adding the same number of voters with the reverse preferences. Then, by positive responsiveness, $F(x) = F(x - x_{D_3})$ as we claimed, and moreover, $F(x) = A$ if and only if $x = x_{D_3} \in D_3$.

Suppose $x, y \in \mathbb{N}^3$ are such that $F(x) \cap F(y) \neq \emptyset$. We can express as $F(x) \cap F(y) = F(x - x_{D_3}) \cap F(y - y_{D_3})$. Let $z = x + y$ and notice that $z_{D_3} = x_{D_3} + y_{D_3}$, since the 'min' operator is additive. Hence, $F(z) = F(x_{D_3} + y_{D_3} + (x - x_{D_3}) + (y - y_{D_3})) = F((x-x_{D_3})+(y-y_{D_3}))$. We claim that $F((x-x_{D_3})+(y-y_{D_3})) = F(x-x_{D_3}) \cap F(y-y_{D_3})$. Notice that if at least one of x, y is in D_3, then the claim is established: if $x \in D_3$, then $x = x_{D_3}$ and $F(x - x_{D_3}) = A$. Suppose $x, y \notin D_3$. Then, $F(x - x_{D_3}) \cap F(y - y_{D_3}) \neq \emptyset$ implies $F(x - x_{D_3}) = F(y - y_{D_3}) = \{a_i\}$ for some $i \in \{1, 2\}$. By positive responsiveness, that is only possible if $\min\{x_1, x_2\} = x_j$ and $\min\{y_1, y_2\} = y_j$ for $j \in \{1, 2\}$ and $j \neq i$. Then by positive responsiveness, $F((x - x_{D_3}) + (y - y_{D_3})) = \{a_i\}$ since one can generate $(x - x_{D_3}) + (y - y_{D_3}) \in \mathbb{N}^3$ from $0 \in D_3$ by adding $(x_i - x_j) + (y_i - y_j)$ many voters with strict preferences of $a_i \succ a_j$. Hence, our second claim is established, which then implies that F is consistent.

Then, the result in part **(a)** implies that F is one of F^*, F_1 and F_2. But none of F^* and F_2 satisfies positive responsiveness. Hence, $F = F_1$ which is the simple majority rule.

2. The proof above is rather indirect and a more direct proof is as follows. We know that anonymity and neutrality imply that, $\forall x \in D_3, F(x) = \{a_1, a_2\}$. We claim that $\forall x' \in D'_3, F(x') = \{a_1\}$. Suppose $x' = (x'_1, x'_2, x'_3) \in D'_3$. Consider $x^* = (x'_2, x'_2, x'_3) \in D_3$. Then, $F(x^*) = \{a_1, a_2\}$. We can generate x' from x^* by adding $(x'_1 - x'_2)$ voters, one at a time, with the preferences of $a_1 \succ a_2$. Then by positive responsiveness, $F(x') = \{a_1\}$. Hence, $\forall x' \in D'_3, F(x') = \{a_1\}$ and by neutrality, $\forall x'' \in D''_3, F(x'') = \{a_2\}$, which imply that $F = F_1$, which is the simple majority rule. ■

May (1952) axiomatizes majority rule with anonymity, neutrality and strong monotonicity. The main difference between May's Theorem and Theorem 4 **(b)** is, May (1952) considers a fixed electorate setting while we consider a variable electorate setting. Then, the axiom of positive responsiveness to voter addition should be seen as a modification of the strong monotonicity axiom to the new setting.[2]

Let us show that one can use cancellation property axiom instead of neutrality in Theorem 3 and 4.

Proposition 2 *Let $m = 2$ and let F be an anonymous ESCF. Then,*

[2]Strictly speaking, positive responsiveness to voter addition can not be stated in the original setting with fixed electorate. But it captures the underlying idea of the strong monotonicity axiom, and hence Theorem 4 **(b)** can be seen as a variant of May's Theorem.

(a) *If F is consistent then it has cancellation property if and only if it is neutral, and*

(b) *If F has cancellation property and is positive responsive then it is consistent.*

Proof. **(a)** IF: We have shown in the proof of Theorem 3 that anonymity, consistency and neutrality imply that, $\forall x \in D_3, F(x) = A$. Since $m = 2$, any profile $x \in \mathbb{N}^3$ is such that the number of voters who prefers $a_1 \gtrsim a_2$ is same as that of who prefers $a_2 \gtrsim a_1$, if and only if $x \in D_3$. Hence, F satisfies the cancellation property.

ONLY IF: Let $F_{|P_0}$ be restriction of F into $P_0 = \{x \in \mathbb{N}^3 : x_1, x_2 \in \mathbb{N}, x_3 = 0\} \subset \mathbb{N}^3$. Note that $F_{|P_0}$ is a SCF and since F satisfies anonymity, consistency and cancellation property, so is $F_{|P_0}$. Then by Proposition 1, we conclude that $F_{|P_0}$ satisfies neutrality. Note also that by cancellation property, $F(0, 0, 1) = A$, which implies that $F(0, 0, x_3) = A$ for all $x_3 \geq 1$, by consistency. Then, we conclude that $\forall x = (x_1, x_2, x_3) \in \mathbb{N}^3$ such that $x_3 \geq 1, F(x_1, x_2, x_3) = F(x_1, x_2, 0) = F_{|P_0}(x_1, x_2)$ since $F(x_1, x_2, x_3) = F(x_1, x_2, 0) + F(0, 0, x_3) = F(x_1, x_2, 0)$, where the last equality follows by consistency. Then, $\forall x \in \mathbb{N}^3, \forall \sigma \in S_2, F \circ \sigma(x) = F_{|P_0} \circ \sigma(x) = \sigma \circ F_{|P_0}(x_1, x_2) = \sigma \circ F(x)$ since the first and the last equality follows by our second conclusion, while the second equality follows by our first conclusion, and hence F is neutral.

(b) We prove the statement indirectly showing that anonymity, cancellation property and positive responsiveness imply that $F = F^M$. By definition, cancellation property implies that $\forall x \in D_3, F(x) = \{a_1, a_2\}$. Repeating the same argument as above in the second proof of Theorem 4 **(b)**, we conclude that, by positive responsiveness, $\forall x' \in D_3', F(x') = \{a_1\}$. Let $x'' = (x_1'', x_2'', x_3'') \in D_3''$. Consider $x^* = (x_1'', x_1'', x_3'') \in D_3$. Then, $F(x^*) = \{a_1, a_2\}$. We can generate x'' from x^* by adding $(x_2'' - x_1'')$ voters, one at a time, with the preferences of $a_2 \succ a_1$. Then by positive responsiveness, $F(x'') = \{a_2\}$. Hence $\forall x'' \in D_3'', F(x'') = \{a_2\}$. So, $F = F^M$ and hence, it is consistent. ■

4. Final comments

When it is presented Young's Theorem is often accompanied by the following remark: "its proof is difficult and omitted" (see for instance, Chap. 9 in Moulin, 1988). However, the above analysis shows that in the special case of voting over two alternatives it can easily be proved. One may also wonder whether the axiom of continuity can be eliminated when there are more than two alternatives. The answer to this question is negative since the example of a (composite) scoring function satisfying the axioms of anonymity, neutrality and consistency but not continuity, given in Sect. 3 of Young (1975), can easily be extended to the case of any finite (but more than three) alternatives. This observation implies that any such elimination contradicts Theorem 1 in Young (1975).

It may seem that the setting with two alternatives is rather restrictive, especially in the context of scoring rules. However, note that the analysis of this simple case can shed a light on possible extensions of some of the axiomatization results in voting theory. For instance, as majority rule and approval voting (AV) coincide when $m = 2$, the result in Theorem 4 **(a)** is related to the axiomatization of AV in Fishburn (1978). In the simple case it is easy to see that the axioms of neutrality and cancellation can be used interchangeably (see Proposition 2). On the other hand, Alos-Ferrer (2006) shows that one can drop neutrality in the presence of anonymity, consistency, cancellation and faithfulness in Fishburn (1978)'s axiomatization of AV. Hence, Alos-Ferrer (2006)'s result can be seen as an extension of Theorem 4 **(a)**. Finally, note also that Theorem 4 **(a)** admits another extension: one can keep neutrality and drop cancellation in the axiomatization of AV, which is, to my best knowledge, an open question.

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