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Symmetry vs. complexity in proving the Muller-Satterthwaite theorem

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Abstract

In this short note, we first provide two rather straightforward proofs for the Muller - Satterthwaite theorem in the baseline cases of 2 person 3 alternatives, and 2 person $n \geq 3$ alternatives. We also show that it suffices to prove the result in the special case of 3 alternatives (with arbitrary N individuals) as it then can easily be extended to the general case. We then prove the result in the decisive case of 3 alternatives (with arbitrary N individuals) by induction on N.

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1. Introduction

An analogous result to Arrow's Impossibility Theorem (Arrow 1963) in the context of voting is the Gibbard-Satterthwaite (G-S) Theorem (Gibbard 1973; Satterthwaite 1975). The interconnection between these two results is a recurrent topic of study in social choice and voting theory. In this respect, it is known that

(a) one can prove each theorem with the help of the other (see Gibbard 1973; Satterthwaite 1975; Schmeidler and Sonnenschein 1978),

(b) one can provide a more general result that implies both theorems (e.g., Miller 2009), and

(c) one can obtain a single proof for both theorems (see Reny 2001).

Moreover, it can be observed that the connections between them are usually obtained through another result, the Muller-Satterthwaite (M-S) Theorem (Muller and Satterthwaite 1977). Hence, the latter constitutes a common ground for the former two. In particular, the fact that the monotonicity axiom in the M-S Theorem is analogous to the independence axiom in Arrow's Impossibility Theorem, and on the unrestricted domain of strict preferences, it is equivalent to the strategy-proofness in the G-S Theorem, allows one to easily obtain results mentioned above in **(a)**-**(c)**: see Reny (2001), Miller (2009) and Chap. 2 in Vohra (2011).

In this paper we first provide two proofs of a variant of the M-S Theorem (Theorem 1, Sect. 2) in the baseline case of 2 person, 3 alternatives. Since it is well known that the M-S Theorem has the G-S Theorem as a corollary (see Reny 2001), we also prove the G-S Theorem in the baseline case. As Barberà (2011) notes, "the 2 person 3 alternative case contains all the essential elements of the (G-S) theorem, in a nutshell," in the sense that it is possible to prove the theorem in the general case by a double induction on the number of individuals and the number of alternatives, once it is proved in the baseline case (see Satterthwaite 1975; Schmeidler and Sonnenschein 1978).

The essence of our proofs is to directly verify the result in the baseline case. However, we reduce the complexity of the problem in two ways: (1) via explicit use of neutrality (symmetry), and (2) via tying up all reasoning on a monotone social choice function with full domain to that of a monotone social choice function with a smaller domain of 1 person society. Then, in Section 4 we show that one can easily prove the M-S Theorem in the general case, once it is proved for the case of 3 alternatives (Proposition 1). Such extension can be useful in inductive proofs of the M-S Theorem. We then complete the proof of the theorem by proving it in the decisive case of 3 alternatives (Proposition 2).

In the next section we introduce our main definitions and state the theorem to be proven. Section 3 gives the proofs of the M-S Theorem in the baseline case, while Section 4 shows how one can extend the M-S Theorem with 3 alternatives to the general case of arbitrary but finite alternatives. The last section concludes.

2. The preliminaries

Let $A = \{a_1, ..., a_n\}$ denote the set of alternatives with $n \in \mathbb{N}$ elements and let X denote the set of strict linear orders (strict rankings) on A. Let there be N individuals in the group. A function $f : X^N \to A$ is called a social choice function (SCF). A member $x = (x_1, ..., x_N)$ of X^N is called a profile of rankings (or simply a profile) and its $i'th$ component, x_i, is called the individual $i's$ ranking. We say that a SCF $f : X^N \to A$ is *Pareto efficient* (PE) if whenever alternative a is on top of x_i for $i = 1, ..., N$, then $f(x) = a$. It is *monotonic* (MT) if whenever $f(x) = a$ and for every individual i and every alternative b the ranking x_i' ranks a above b if x_i does, then $f(x') = a$. Finally, it is *dictatorial* (DT) if there is individual i such that $f(x) = a$ if and only if a is at the top of x_i, and we denote such function as f_d^i, for $i = 1, ..., N$.

The following result is known as (a variant of) the M-S Theorem (see also Reny 2001):

Theorem 1 *If $n \geq 3$, a SCF $f : X^N \to A$ is PE and MT if and only if it is DT.*

3. The proofs for the baseline case

Let us introduce a binary relation R_{a_1} on X, called as the *monotonicity relation w.r.t a_1 on* X, defined as $\forall x, y \in X$, $xR_{a_1}y$, i.e. x is related to y according to R_{a_1}, if $x, y \in X$ are such that for any alternative a_j, if a_1 is ranked above a_j in x then so is in y. R_{a_i} for $i = 2, ..., n$ are defined analogously. Whenever $xR_{a_i}y$, we say y is a *successor* of x in R_{a_i}. We also introduce a similar binary relation ($R_{a_1}^N$) on the full domain of X^N: $\forall x, y \in X^N$, $xR_{a_1}^N y$ if $x, y \in X^N$ are such that for any alternative a_j and any individual $i = 1, ..., N$, if a_1 is ranked above a_j in x_i, then so is in y_i. $R_{a_i}^N$ for $i = 2, ..., n$ are defined analogously.[1] Whenever $xR_{a_i}^N y$, we say y is a *successor* of x in $R_{a_i}^N$. Note that by definition, for any MT SCF $f : X^N \to A$, $xR_{a_i}^N y$ implies that if $f(x) = a_i$, then $f(y) = a_i$, for $i = 1, ..., n$.

$R_{a_i}^N$ for $i = 1, ..., n$ has the following properties:

Lemma 1 *For $i = 1, ..., n$,*

(a) $\forall x, y \in X^N, xR_{a_i}^N y$ *if and only if* $x_j R_{a_i} y_j$*, for* $j = 1, ..., N$*, and*

(b) $R_{a_i}^N$ *is a preorder (reflexive and transitive) on* X^N*.*

Proof. **(a)** Both directions of the statement immediately follow from the definitions of R_{a_i} and $R_{a_i}^N$. **(b)** Observe that it is easy to verify that R_{a_i} is a preorder on X. The result follows combining this observation with **(a)**. ■

Lemma 1 **(a)** allows any reasoning on $R_{a_i}^N$ to be entirely based on R_{a_i}, for $i = 1, ..., n$, while Lemma 1 **(b)** allows us to reason recursively.

Let us now assume $N = 2$ and $n = 3$. We can code the elements of X as follows: $a_1 \succ a_2 \succ a_3 \equiv 123$; $a_1 \succ a_3 \succ a_2 \equiv 132$; $a_2 \succ a_1 \succ a_3 \equiv 213$; $a_2 \succ a_3 \succ a_1 \equiv 231$; $a_3 \succ a_1 \succ a_2 \equiv 312$; $a_3 \succ a_2 \succ a_1 \equiv 321$. Let's construct the following graph which represents R_{a_1}:

$$\begin{array}{ccccccc} & & & (213) & & & \\ (321) & \longleftrightarrow & (231) \begin{array}{c}\nearrow\\ \searrow\end{array} & & \begin{array}{c}\searrow\\ \nearrow\end{array}(132) & \longleftrightarrow & (123) \\ & & & (312) & & & \end{array} \qquad (\Gamma_{a_1})$$

We call Γ_{a_1} as "Monotonicity Graph for a_1" and note that $\forall \alpha, \beta \in V(\Gamma_{a_1})$ (the set of vertices), $\alpha R_{a_1} \beta$ if and only if there is directed path from α to β (we assume that every

[1] See also the notion of *monotonic transformation* in Klaus and Bochet (2011).

node is connected to itself by a directed path). Note that if a node is assigned to a_1 under any MT SCF $g : X \to A$, then all of its successors in Γ_{a_1} must be assigned to a_1. Similarly, we can create 'monotonicity graphs' for a_2 and a_3:

$$\begin{array}{ccccccccc} & & & & (123) & & & & \\ (312) & \longleftrightarrow & (132) & \begin{array}{c}\nearrow\\ \searrow\end{array} & & \begin{array}{c}\searrow\\ \nearrow\end{array} & (231) & \longleftrightarrow & (213) \\ & & & & (321) & & & & \end{array} \qquad (\Gamma_{a_2})$$

$$\begin{array}{ccccccccc} & & & & (231) & & & & \\ (123) & \longleftrightarrow & (213) & \begin{array}{c}\nearrow\\ \searrow\end{array} & & \begin{array}{c}\searrow\\ \nearrow\end{array} & (312) & \longleftrightarrow & (321) \\ & & & & (132) & & & & \end{array} \qquad (\Gamma_{a_3})$$

Proof 1: Let us now prove the M-S Theorem. Let $f : X^2 \to A$ be a MT and PE SCF and for $i = 1, 2, 3$, let D_i denote the set of profiles such that a_i is ranked at the top of each ranking:

$$D_1 = \{(123, 123), (123, 132), (132, 123), (132, 132)\},$$

$$D_2 = \{(213, 213), (213, 231), (231, 213), (231, 231)\},$$
$$D_3 = \{(312, 312), (312, 321), (321, 312), (321, 321)\}.$$

Note that none of the profiles in $D_2 \cup D_3$ can be assigned to a_1, by PE. Hence, none of their *predecessors* in $R^2_{a_1}$ (α is a predecessor of β if $\alpha R^2_{a_1} \beta$ and $\alpha \neq \beta$) can be assigned to a_1. So, every profile in

$$P(D_2 \cup D_3) = \{(213, 321), (321, 213), (231, 312), (312, 231), (231, 321), (321, 231)\}$$

needs to be assigned either to a_2 or a_3. Let $f(213, 321) = a_3$. Then, referring to $R^2_{a_3}$ (to Γ_{a_3}) we conclude that

$$\begin{aligned} \omega_1 \quad &: \quad f(213, 321) = f(231, 321) = f(231, 312) = f(213, 312) = \\ &= \quad f(123, 312) = f(132, 312) = f(123, 321) = f(132, 321) = a_3. \end{aligned}$$

Note that there is a complete *symmetry* among the elements of A in our renaming them as a_1, a_2 and a_3: any of the a_1, a_2, a_3 can equally represent any of the three alternatives in A. This *symmetry* is often called as the *neutrality axiom* and it is implicit in our definition of SCF. Because of the *symmetry* between a_3 and a_2 (exchanging the roles of a_3 and a_2), we can conclude that decisions in ω_1 are one and the same as the following decisions:

$$\begin{aligned} \omega_2 \quad &: \quad f(132, 213) = f(123, 213) = f(123, 231) = f(132, 231) = \\ &= \quad f(312, 231) = f(321, 231) = f(312, 213) = f(321, 213) = a_2. \end{aligned}$$

Similarly, since there is a *symmetry* between a_3 and a_1, they are are also one and the same as the following decisions:

$$\begin{aligned} \omega_3 \quad &: \quad f(231, 123) = f(213, 132) = f(213, 123) = f(231, 132) = \\ &= \quad f(321, 132) = f(312, 132) = f(321, 123) = f(312, 123) = a_1. \end{aligned}$$

Hence, **once the initial decision is made, all the other decisions follow** (recall that the profiles in D_i are assigned to a_i by PE, for $i = 1, 2, 3$). Alternatively, let $f(213, 321) = a_2$. Note that $f(213, 132) \neq a_3$, since otherwise referring to $R^2_{a_3}$ we conclude that $f(213, 321) = a_3$, which is a contradiction. Note also that $f(213, 132) \neq a_1$ since otherwise by the *symmetry* between a_1 and a_3, we conclude that $f(231, 312) = a_3$, which then implies that (referring to $R^2_{a_3}$) $f(231, 321) = a_3$. But $(231, 321)$ is a successor of $(213, 321)$ in $R^2_{a_2}$, hence $f(231, 321) = a_2$, which is a contradiction. So, $f(213, 132) = a_2$. Then referring to $R^2_{a_2}$, we can conclude that

$$\begin{aligned} \varphi_1 \quad &: \quad f(213,321) = f(231,321) = f(213,132) = f(213,123) = \\ &= \quad f(231,132) = f(231,123) = f(231,312) = f(213,312) = a_2. \end{aligned}$$

Because of *symmetry*, decisions in φ_1 are one and the same as the following decisions:

$$\begin{aligned} \varphi_2 \quad &: \quad f(123,312) = f(132,312) = f(123,231) = f(123,213) = \\ &= \quad f(132,231) = f(132,213) = f(132,321) = f(123,321) = a_1 \end{aligned}$$

and

$$\begin{aligned} \varphi_3 \quad &: \quad f(321,132) = f(312,132) = f(321,213) = f(321,231) = \\ &= \quad f(312,213) = f(312,231) = f(312,123) = f(321,123) = a_3. \end{aligned}$$

Hence, there are only two possible assignments, $\{\omega_1, \omega_2, \omega_3\}$ and $\{\varphi_1, \varphi_2, \varphi_3\}$, and each of them corresponds to a DT social choice function with one of the two individuals being a dictator. This completes the proof.

Proof 2: Suppose $f(213, 321) = a_3$. Note that (213) has 6 successors in Γ_{a_3} while (321) has 2. Since by Lemma 1 **(a)** any combination of successors of (213) and (321) in Γ_{a_3} is a successor of $(213, 321)$ in $R^2_{a_3}$, there are $12 = 6 \times 2$ (including $(213, 321)$) profiles to be assigned to a_3. By *symmetry*, then there are 12 profiles to be assigned to a_i, $i = 1, 2$. Since X^2 has 36 elements, once the initial decision is made all the other decisions follow i.e., there is a unique function $f : X^2 \to A$ which is PE, MT and $f(213, 321) = a_3$. On the other hand $f^2_d : X^2 \to A$ has these properties: it is PE, MT and $f^2_d(213, 321) = a_3$. Hence, $f = f^2_d$.

Alternatively, suppose $f(213, 321) = a_2$. Then, by the same argument as in Proof **1** we conclude that $f(213, 132) = a_2$. Then, repeating the same argument just used for the case of $f(213, 321) = a_3$, we conclude that there is a unique PE and MT function $f : X^2 \to A$ such that $f(213, 132) = a_2$. Since $f^1_d : X^2 \to A$ has these properties, we then conclude that $f = f^1_d$. This completes our proof.

Proof 2 for $N = 2$, $n \geq 3$: Consider a profile $x \in X^2$ such that $x_1 = (a_1 \succ a_2 \succ ... \succ a_n)$ and $x_2 = (a_2 \succ a_3 \succ ... \succ a_n \succ a_1)$. Let $f : X^2 \to A$ be PE and MT. We claim that $f(x) \in \{a_1, a_2\}$. Suppose on the contrary that $f(x) = a_j \notin \{a_1, a_2\}$. Then, consider $x^*_1 = (a_2 \succ a_1 \succ a_3 \succ ... \succ a_n)$. By MT, if $f(x) = a_j \notin \{a_1, a_2\}$, then $f(x^*_1, x_2) = a_j$ which then contradicts PE. Hence, the claim is established. Let $f(x) = a_1$. Since any ranking is a successor of x_2 in R_{a_1}, it has $n!$ successors, and since any ranking with a_1 ranked at the top is a successor of x_1 in R_{a_1}, it has $(n-1)!$ successors. Then by Lemma 1 **(a)**, any combination of successors of x_1 and x_2 in R_{a_1} is a successor of x in $R^2_{a_1}$, there are $n!(n-1)!$ profiles to be assigned to a_1 under f. By *symmetry*, then there are $n!(n-1)!$ profiles to be assigned to a_i, $i = 2, ..., n$. Since X^2 has $(n!)^2$ elements, once the initial decision is made all the other decisions follow. Hence, there is a unique PE and MT $f : X^2 \to A$ with $f(x) = a_1$. But since f_d^1 has these properties, we conclude that $f = f_d^1$.

Alternatively, let $f(x) = a_2$. Let $x' \in X^2$ be such that $x'_1 = x_1 = (a_1 \succ a_2 \succ ... \succ a_n)$ and $x'_2 = (a_2 \succ a_1 \succ a_3 \succ ... \succ a_n)$. Since $f(x) = a_2$, $f(x') = a_2$ by MT. Consider $x'' \in X^2$ such that $x''_1 = (a_1 \succ a_3 \succ ... \succ a_n \succ a_2)$ and $x''_2 = x'_2$. We claim that $f(x'') \in \{a_1, a_2\}$. Suppose on the contrary $f(x'') = a_j \notin \{a_1, a_2\}$. Consider $x^*_2 = (a_1 \succ a_2 \succ a_3 \succ ... \succ a_n)$. If $f(x'') = a_j \notin \{a_1, a_2\}$, then $f(x'_1, x^*_2) = a_j$ by MT, which then contradicts PE. Hence, the claim is established. Note that $f(x'') \neq a_1$ as otherwise it would imply that $f(x') = a_1$ by MT. Hence, we conclude that $f(x'') = a_2$. Then by the same argument as above we can show that there is a unique PE and MT function with $f(x'') = a_2$, and since f_d^2 has these properties, we then conclude that $f = f_d^2$.

4. Sufficiency of proving the M-S Theorem for $n = 3$

The following result shows that it suffices to prove Theorem 1 when $n = 3$:

Proposition 1 *Suppose Theorem* 1 *holds when* $n = 3$. *Then it holds for any finite* $n > 3$.

Proof. Let $n > 3$ and let $f : X^N \to A$ be a MT and PE SCF. Let $A_3 = \{a_1, a_2, a_3\} \subset A$ and let $X_{A_3} \subset X^N$ be the set of all profiles $x \in X^N$ such that for each x_i, $i = 1, ..., N$, the top 3 alternatives of x_i are in A_3, and for $j = 4, ..., n$, the $j'th$ top alternative of x_i is $a_j \in A$. We claim that $\forall x \in X_{A_3}$, $f(x) \in A_3$. Suppose on the contrary that $\exists y \in X_{A_3}$ such that $f(y) = a_r$ with $r > 3$. By MT this implies that $\forall x \in X_{A_3}$, $f(x) = a_r$ which contradicts PE. Hence, the claim is established.

Let X_3 be the set of all strict rankings on A_3 and let us define $f^3 : X_3^N \to A_3$ as $\forall z \in X_3^N$, $f^3(z) = f(x^z)$ where $x^z \in X_{A_3}$ is a profile such that x_i^z and z_i coincide on A_3, i.e. $x_i^z = (z_i \succ a_4 \succ ... \succ a_n)$, for all $i = 1, ..., N$. Notice that for each $z \in X_3^N$ there is a unique such $x^z \in X_{A_3}$. Combining this with our claim we conclude that, f^3 is a well defined 3 alternative SCF. Moreover, since f is PE and MT, so is f^3. Hence, by our hypothesis f^3 must be DT.

Without loss of generality we may assume that 1 is the dictator of f^3. Let us then show that 1 is the dictator of f. Consider $z \in X_3^N$ such that $z_1 = (a_1 \succ a_2 \succ a_3)$ and $z_i = (a_2 \succ a_1 \succ a_3)$ for $i = 2, ..., N$. Since 1 is the dictator of f^3, $f^3(z) = f(x^z) = a_1$. Now let $x'_i = (a_2 \succ a_3 \succ ... \succ a_n \succ a_1)$ for $i = 2, ..., N$. We first claim that $f(x_1^z, x'_2, x_3^z, ..., x_N^z) = a_1$. Note that $f(x_1^z, x'_2, x_3^z, ..., x_N^z) \neq a_2$ as otherwise it would imply that $f(x^z) = a_2$ by MT, and also $f(x_1^z, x'_2, x_3^z, ..., x_N^z) \notin \{a_3, ..., a_n\}$ since if $f(x_1^z, x'_2, x_3^z, ..., x_N^z) = a_j$ for some $j \in \{3, ..., n\}$, then $f(x'_1, x'_2, x_3^z, ..., x_N^z) = a_j$ where $x'_1 = (a_2 \succ a_1 \succ a_3 \succ ... \succ a_n)$ by MT, which then contradicts PE. Hence, $f(x_1^z, x'_2, x_3^z, ..., x_N^z) = a_1$.

We can change rankings of individuals 3 to N from x_i^z to x'_i, each at a time, and repeat the same argument to conclude that $f(x_1^z, x'_2, ..., x'_N) = a_1$. Notice that x_1^z has $(n-1)!$ successors in R_{a_1}, while x'_i has $n!$ successors in R_{a_1}. By Lemma 1 **(a)**, then $(x_1^z, x'_2, x'_3, ..., x'_N) \in X^N$ has $(n-1)!(n!)^{N-1}$ successors in $R_{a_1}^N$. Hence, there are $(n-1)!(n!)^{N-1}$ profiles to be assigned to a_1 under f. By *symmetry*, then there are $(n-1)!(n!)^{N-1}$ profiles to be assigned to a_i, $i = 2, ..., n$. Since X^N has $(n!)^N$ elements, there is a unique PE and MT SCF such that $f(x_1^z, x_2', ..., x_N') = a_1$. But since $f_d^1 : X^N \rightarrow A$ has these properties, we conclude that $f = f_d^1$. ■

For completeness, let us verify that Theorem 1 holds when $n = 3$.

Proposition 2 *Theorem* 1 *holds when* $n = 3$.

Proof. We use induction on N. As shown above in Proof **1** and **2**, the statement is true when $N = 2$. Suppose it is true when $N = k \geq 2$ and let us consider the case of $N = k+1$. Let $f : X^{k+1} \rightarrow A$ be PE and MT SCF. Consider a profile $x \in X^{k+1}$ with $x_1 = (a_1 \succ a_2 \succ a_3)$ and $x_i = (a_2 \succ a_3 \succ a_1)$ for $i = 2, ..., k+1$. Then, $f(x) \neq a_3$ since otherwise by MT $f(x^*) = a_3$ for $x^* \in X^{k+1}$ such that $x_1^* = (a_2 \succ a_1 \succ a_3)$ and $x_i^* = x_i$, $i = 2, ..., k+1$, which then contradicts PE. Hence, $f(x) \in \{a_1, a_2\}$. Suppose $f(x) = a_1$. Notice that $x_1 \in X$ has 2 successors in R_{a_1} while $x_i \in X$ has 3! successors in R_{a_1}. By Lemma 1 **(a)**, then $x \in X^{k+1}$ has $2 \cdot (3!)^k$ successors in $R_{a_1}^{k+1}$, and there are $2 \cdot (3!)^k$ many profiles to be assigned to a_1 under f. By *symmetry*, then there are $2 \cdot (3!)^k$ many profiles to be assigned to a_i, $i = 2, 3$. Since X^{k+1} has $(3!)^{k+1}$ elements, there is a unique $f : X^{k+1} \rightarrow A$ which is PE, MT and satisfies $f(x) = a_1$. Since f_d^1 has these properties, we conclude that $f = f_d^1$.

Alternatively, suppose $f(x) = a_2$. Let us define $g : X^k \rightarrow A$ as $\forall y \in X^k$, $g(y) = f(x_1, y_2, ..., y_{k+1})$, i.e. we fix individual $1's$ ranking at x_1. Note that since f is MT, so is g. We claim that g is also PE. Notice that when $a_1 \in A$ is on top of each ranking y_i, $i = 2, ..., k+1$, $g(y) = a_1$, by PE of f. Note also that when $a_2 \in A$ is on top of each y_i, $i = 2, ..., k+1$, $g(y) = a_2$ by MT. Consider $x' \in X^{k+1}$ such that $x_1' = (a_1 \succ a_3 \succ a_2)$ and $x_i' = (a_3 \succ a_2 \succ a_1)$ for $i = 2, ..., k+1$. Then, $f(x') \neq a_1$ since otherwise $f(x) = a_1$ by MT, which is a contradiction. Also $f(x') \neq a_2$ since otherwise $f(x^{**}) = a_2$ for $x^{**} \in X^{k+1}$ such that $x_1^{**} = (a_3 \succ a_1 \succ a_2)$ and $x_i^{**} = x_i'$, $i = 2, ..., k+1$, which then contradicts PE. Hence, $f(x') = a_3$. By MT, this implies that $f(x'') = a_3$ for $x'' \in X^{k+1}$ such that $x_1'' = x_1'$ and $x_i'' = (a_3 \succ a_1 \succ a_2)$ for $i = 2, ..., k+1$.

Consider $x''' \in X^{k+1}$ such that $x_1''' = x_1 = (a_1 \succ a_2 \succ a_3)$ and $x_i''' = x_i''$, $i = 2, ..., k+1$. Then, $f(x''') \neq a_1$ since otherwise $f(x'') = a_1$ by MT, which is a contradiction as we just concluded that $f(x'') = a_3$. Also $f(x''') \neq a_2$ since otherwise by MT $f(x^{***}) = a_2$ for $x^{***} \in X^{k+1}$ such that $x_1^{***} = x_1'''$ and $x_i^{***} = (a_1 \succ a_3 \succ a_2)$ for $i = 2, ..., k+1$, which then contradicts PE. Hence, we conclude that $f(x''') = a_3$. This implies that, for all $y \in X^k$ such that a_3 is ranked at the top of each y_i, $i = 2, ..., k+1$, $g(y) = f(x_1, y_2 ..., y_{k+1}) = a_3$. Hence, $g : X^k \rightarrow A$ is PE.

Then by our induction hypothesis, $g : X^k \rightarrow A$ is DT. Without loss of generality, we may assume that individual 2 is the dictator of g. We claim that 2 is also the dictator of f. Consider $x^{\tilde{}} \in X^{k+1}$ such that $x_2^{\tilde{}} = (a_3 \succ a_2 \succ a_1)$ and $x_i^{\tilde{}} = x_1$ for $i = 1, 3, ..., k+1$. Then, $f(x^{\tilde{}}) = g(x_2^{\tilde{}}, ..., x_{k+1}^{\tilde{}}) = a_3$. Repeating the same argument as in the first part of the proof, we can conclude that there is a unique $f : X^{k+1} \rightarrow A$ which is PE, MT and $f(x^{\tilde{}}) = a_3$. Since f_d^2 has these properties, we conclude that $f = f_d^2$. This completes our proof. ■

5. Final comments

In the first part of this paper (Sect. 3), we presented two rather straightforward proofs of the Muller-Satterthwaite (M-S) Theorem in the baseline case of 2 person and 3 alternatives. With a slight modification of the set up each approach can prove Arrow's Impossibility Theorem in that case. Moreover, in principle it is possible to prove the M-S Theorem in the general case using the same approach. In order to that, one needs to investigate more abstract properties of the binary relations introduced above. In particular, the fact that these relations can be defined recursively, starting with the simplest case of a single individual profile, indicates a possibility for such investigation.

In the second part (Sect. 4), we showed how one can extend the special case of the M-S Theorem with 3 alternatives to the general case of arbitrary but finite alternatives (in Proposition 1). Such extension can be relevant for inductive proofs of the M-S Theorem. In particular, it shows that for such proofs, using double induction on both number of alternatives and number individuals is unnecessary.

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