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The fractional integrated bi- parameter smooth transition autoregressive model

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Abstract

This paper introduces the fractionally integrated Bi-parameter smooth transition autoregressive model (FI-BSTAR model) as an extension of BSTAR model proposed by Siliverstovs (2005) and the fractionally integrated STAR model (FI-STAR model) proposed by van Dijk et al. (2002). Our FI-BSTAR model is able to simultaneously describe persistence and asymmetric smooth structural change in time series. An empirical application using monthly growth rates of the American producer price index is provided.

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1 Introduction

Time series methods, whether parametric or nonparametric, has established a broad field of knowledge whose application stretches over both natural and social sciences. Within this valuable area, a fast growing research gave rise to important developments enriching the use of time series econometrics in empirical applications. Along this progress, we continously observe some theoretical and empirical findings pertaining to long memory concepts. Surprisingly, the notion of long memory has never been completely agreed upon. However, statisticians unanimously argue that long memory or long range dependence means that observations far away from each other are still strongly correlated. Accordingly, correlations of long memory processes decay slowly with a hyperbolic rate. Also, long range dependence implies that the present information has a persistent impact on future counts. Furthermore, the presence of long memory dynamics cause nonlinear dependence in the first moment of the distribution and hence acts as a potentially predictable component in the series dynamics. Readers are referred to Granger and Joyeux (1980) and Hosking (1981) for the main theoretical contributions.

Nonlinearity is another key property that coexists with long memory. A natural approach to modeling economic time series with nonlinear models is used to define different states of the world or regimes, and to allow for the possibility that the dynamic behavior of economic variables depends on the regime that occurs at any given point in time. However, there are two main regime switching models: the so-called Smooth Transition Regression model (STR model) and the popular Markov-Switching model proposed by Hamilton (1989).

Several studies have explored the two key properties of economic and financial time series, namely long-memory and nonlinear properties. Indeed, the theory recently proposed what can be called "nonlinear long-memory" models (see van Dijk et al. 2002, and Ajmi et al. 2008). Subsequently, fractionally integrated smooth transition autoregressive ($FISTAR$) models have also been proposed (see, inter alia, van Dijk et al. 2002, and Smallwood 2005). van Dijk et al. (2002) present the modelling cycles for specifying these models combining the concepts of fractional integration and smooth transition nonlinearity for the US unemployment rate.

Our work fits in the above-mentioned field of research. We propose an extension of the Bi-parameter smooth transition autoregressive model ($BSTAR$ model) proposed by Siliverstovs (2005) as a generalization of the $LSTR2$ model suggested earlier in Teräsvirta (1998). The $BSTAR$ model suggested a Bi-parameter transition function having two slopes and two threshold parameters allowing for different transition speeds between middle and outer regimes. More specifically, we introduce a new model; the fractionally integrated $BSTAR$ model ($FI-BSTAR$ model), able to allow both for structural change, as described by Siliverstovs (2005) in his $BSTAR$ model and used by Ajmi and El Montasser (2012) in their $SEA-BSTAR$ model, and long memory properties inspired from the fractional integrated $STAR$ model ($FI-STAR$) proposed by van Dijk et al. (2002).

Then, the paper is organized as follows. In section 2, we introduce the fractionally integrated Bi-parameter smooth transition autoregressive model ($FI-BSTAR$). In section 3, we empirically specify our $FI-BSTAR$ model based on the method proposed by Teräsvirta (1994) for the basic $STAR$ model. In section 4, the model is empirically fitted to monthly growth rates of the American producer price index. Finally, section 5 concludes. The appendix gives some derivation details.

2 The fractionally integrated BSTAR model

The fractionally integrated Bi-parameter smooth transition autoregressive model ($FI-BSTAR$) is an extension of $BSTAR$ introduced by Siliverstovs (2005). Our modification consists in adding a fractional integration parameter, i.e, (see Granger and Joyeux 1980) to have a model able to describe long memory and asymmetric nonlinearity in time series.

The long memory $BSTAR$ model is given by:

$$(1-L)^d y_t = x_t \tag{1}$$

with

$$x_t = \phi_0 + \sum_{i=1}^{p} \phi_i x_{t-i} + \left(\theta_0 + \sum_{i=1}^{p} \theta_i x_{t-i}\right) \times F(\gamma_1, \gamma_2, c_1, c_2; y_{t-z}) + \varepsilon_t \tag{2}$$

with d is the fractional integration degree of the process. L is the backshift operator such that $Ly_t = y_{t-1}$. $\phi = (\phi_0, \cdots, \phi_p)'$ and $\theta = (\theta_0, \cdots, \theta_p)'$ are autoregressive parameters, respectively, in the first and second regime and $\varepsilon_t \sim NID(0, \sigma^2)$. $F(\gamma_1, \gamma_2, c_1, c_2; y_{t-z})$ is a Bi-parameter transition function characterized by the asymmetric transition function which implies different local dynamics in the neighborhood of the respective location parameters, which is written as follows:

$$\begin{aligned} F(\gamma_1, \gamma_2, c_1, c_2; y_{t-z}) &= \frac{\exp[-\gamma_1 (y_{t-z} - c_1)] + \exp[\gamma_2 (y_{t-z} - c_2)]}{1 + \exp[-\gamma_1 (y_{t-z} - c_1)] + \exp[\gamma_2 (y_{t-z} - c_2)]} \\ \gamma_1 &> 0,\ \gamma_2 > 0,\ c_1 < c_2 \end{aligned}$$

γ_1 and γ_2 are two slope parameters, c_1 and c_2 are two threshold parameters and y_{t-z} is the transition variable. This function is a generalization of the $LSTR2$ model (Terasvirta 1998) and the $AESTAR$ model (Anderson 1997) and guarantees asymmetric transition speed from the outer-lower regime to the middle and from the middle to the outer-higher regime. If $\gamma_1 = \gamma_2 = \gamma$, the Bi-parameter transition function closely approximates the $LSTR2$ transition function, mainly for large values of the slope parameter. When $\gamma_1 \longrightarrow \infty$ and $\gamma_2 \longrightarrow \infty$, $F(\gamma_1, \gamma_2, c_1, c_2; y_{t-z}) \longrightarrow 0$ for $c_1 \leq y_{t-z} \leq c_2$ and $F(\gamma_1, \gamma_2, c_1, c_2; y_{t-z}) \longrightarrow 1$ otherwise[1].

3 The empirical specification of the fractionally integrated $BSTAR$ model

The empirical specification of our fractionally integrated $BSTAR$ is based on the strategy proposed by Granger (1993), i.e., a "specific-to-general" procedure specific to nonlinear time series models. We extend the empirical procedure used by Teräsvirta (1994) for $STAR$ models, van Dijk et al. (2002) for $FISTAR$ models and Siliverstovs (2005) for $BSTAR$ models to elaborate an empirical specification for fractionally integrated $BSTAR$ models.

The specification of $FI-BSTAR$ models consists of the following steps:

1. Specify an appropriate autoregressive order p for a $ARFI$ model using the BIC criterion.

[1] For more details, see Siliverstovs (2005)

2. Test the null hypothesis of linearity against the alternative of Long memory $BSTAR$ nonlinearity and select the appropriate transition variables.

3. Estimate the parameters of our $FI-BSTAR$ model.

4. Evaluate the model using diagnostic tests.

3.1 Nonlinearity test

Having specified $AR(p)$ for a given time series, we proceed by testing nonlinearity using a redefined transition function $F_t^*(.) = F_t(.) - 2/3$.[2]

The model presented in equations (1) and (2) is linear when the slope parameters in both transition functions are equal to zero, i.e., $H_0 : \gamma_1 = \gamma_2 = 0$. We clearly see that our model in equation (1) is not identified under the null hypothesis. For circumventing this problem, we replace the transition functions $F_t^*(.)$ in equations (1) and (2) by their Taylor expansion around the point $\gamma_1 = \gamma_2 = 0$ as proposed by Luukkonen et al. (1988) .

After substituting the first-order Taylor series approximation for $F_t^*(.)$ in equation (1) and (2) and rearranging terms, we get the auxiliary regression:

$$x_t = \alpha_0 + \alpha_1^{'} w_t + \alpha_2^{'} w_t y_{t-z} + e_t \tag{3}$$

where α_0 is a constant, $w_t = (x_{t-1}, ..., x_{t-p})$ and e_t is the residual terms such that under H_0, $e_t = \varepsilon_t$.

As noted by Luukkonen et al. (1988), the nonlinearity LM test based on auxiliary regression (3) is powerless in situations where only the intercept is different across regimes. To remedy this problem, Luukonen et al (1988) suggest a higher-order Taylor expansion.

Replacing the transition function $F_t^*(.)$ with its second-order Taylor approximation yields the following auxiliary regression model

$$x_t = \alpha_0 + \alpha_1^{'} w_t + \alpha_2^{'} w_t y_{t-z} + \alpha_3^{'} w_t y_{t-z}^2 + e_t \tag{4}$$

where α_0, e_t and w_t are presented above.

By assuming the normal distribution of errors, the conditional log-likelihood for observation t is written as:

$$\ln l_t = -\frac{1}{2} \ln (2\pi) - \frac{1}{2} \ln \left(\sigma^2\right) - \frac{e_t^2}{2\sigma^2}$$

Under the linearity hypothesis H_0, the remaining partial derivatives are given by:

$$\left.\frac{\partial \ln l_t}{\partial \alpha_i}\right|_{H_0} = \frac{1}{\sigma^2} \hat{\varepsilon}_t w_t y_{t-z}^i, \quad i = 0, 1, 2.$$

$$\left.\frac{\partial \ln l_t}{\partial d}\right|_{H_0} = -\frac{1}{\sigma^2} \hat{\varepsilon}_t \sum_{j=1}^{t-1} \frac{\hat{\varepsilon}_{t-j}}{j}$$

The LM test based on regression (4) is conducted through the following steps:

1. Estimate the $ARFI$ model and we calculate the residuals $\hat{\varepsilon}_t$ and the sum of squared residuals $SSR_0 = \sum_{t=1}^{T} \hat{\varepsilon}_t^2$.

[2] $F_t^*(.)$ takes a zero under the null hypothesis of linearity.

2. Estimate the auxiliary regression of $\hat{\varepsilon}_t$ on $w_t y^i_{t-z}$ and $-\sum_{j=1}^{t-1} \frac{\hat{\varepsilon}_{t-j}}{j}$, $i = 0, 1, 2$ and compute the sum of squared residuals from this regression $SSR_1 = \sum_{t=1}^{T} \hat{e}_t^2$.

3. Calculate the LM statistic as:

$$LM = \frac{(SSR_0 - SSR_1)/df_1}{SSR_1/df_2} \rightsquigarrow F(df_1, df_2)$$

with $df_1 = 2p$ and $df_2 = T - 3p - 1$.

3.2 Estimation

When the transition variable is selected from the nonlinearity test, the next stage of specification procedure consists in estimating the parameters in the fractionally integrated $BSTAR$ models.

Our $FI - BSTAR$ is estimated using the maximum likelihood method. The numerical solution to the iterative estimation procedure can be obtained using Berndt, Hall, Hall and Hausman (1974) ($BHHH$) algorithm. The $BHHH$ is implemented using the analytical derivatives of the corresponding likelihood functions.

Assuming the normality of the error term, the log-likelihood of the model for one observation is:

$$\ln l_t = -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln\left(\sigma^2\right) - \frac{e_t^2}{2\sigma^2},\ t = 1, \cdots, T.$$

where $e_t = x_t - \phi_0 - \sum_{i=1}^{p} \phi_i x_{t-i} - (\theta_0 + \sum_{i=1}^{p} \theta_i x_{t-i}) \times F(\gamma_1, \gamma_2, c_1, c_2; y_{t-z})$ for the $FI - BSTAR$ in equation (1) and (2).

The partial derivatives of the log-likelihood function necessary to calculate the $BHHH$ approximation to the information matrix, with respect to the $FI - BSTAR$ model in equation (1) and (2), are presented in Appendix 1.

3.3 Misspecification Test

After testing nonlinearity and estimating the parameters, the next step consists in evaluating the fitted $FI - BSTAR$ by testing the residuals serial correlation.

In this section, we present the LM approach for testing the serial correlation for a fractionally integrated $BSTAR$ based on Eirtheim and Teräsvirta's (1996) misspecification of $STAR$ models.

The $FI - BSTAR$ in equation (1) is given by:

$$x_t = H(w_t, \Psi) + \varepsilon_t$$

with $H(w_t, \Psi)$ is the skeleton of the model defined by $\phi_0 + \sum_{i=1}^{p} \phi_i x_{t-i} + (\theta_0 + \sum_{i=1}^{p} \theta_i x_{t-i}) \times F_t(\gamma_1, \gamma_2, c_1, c_2; y_{t-z})$ and $\Psi = \left(\phi', \theta', \gamma_1, \gamma_2, c_1, c_2, d\right)'$.

The no residual correlation LM test for $FI - BSTAR$ is given by these steps:

1. Estimate the $FI - BSTAR$ in equation (1) and (2) and calculate the residuals $\hat{\varepsilon}_t$ and the sum of squared residuals $SSR_0 = \sum_{t=1}^{T} \hat{\varepsilon}_t^2$.

2. Regress $\hat{\varepsilon}_t$ on $(\hat{\varepsilon}_{t-1}, \cdots, \hat{\varepsilon}_{t-q})'$ and $\partial H\left(w_t, \hat{\Psi}\right)/\partial\Psi$ and compute the sum of squared residuals from this regression SSR with q as the serial dependence order[3].

3. Calculate the LM statistic as:

$$LM = \frac{(SSR_0 - SSR)/q}{SSR/T - n - q} \rightsquigarrow F(q, T - n - q)$$

with $n = 2p + 7$.

4 Empirical Application

4.1 Data

This study makes use of the growth rates of the American producer price index, displayed in Figure 1, as an empirical illustration of the suggested model. In this paper we consider seasonally adjusted monthly data covering the period from 1947:4 to 2011:5.

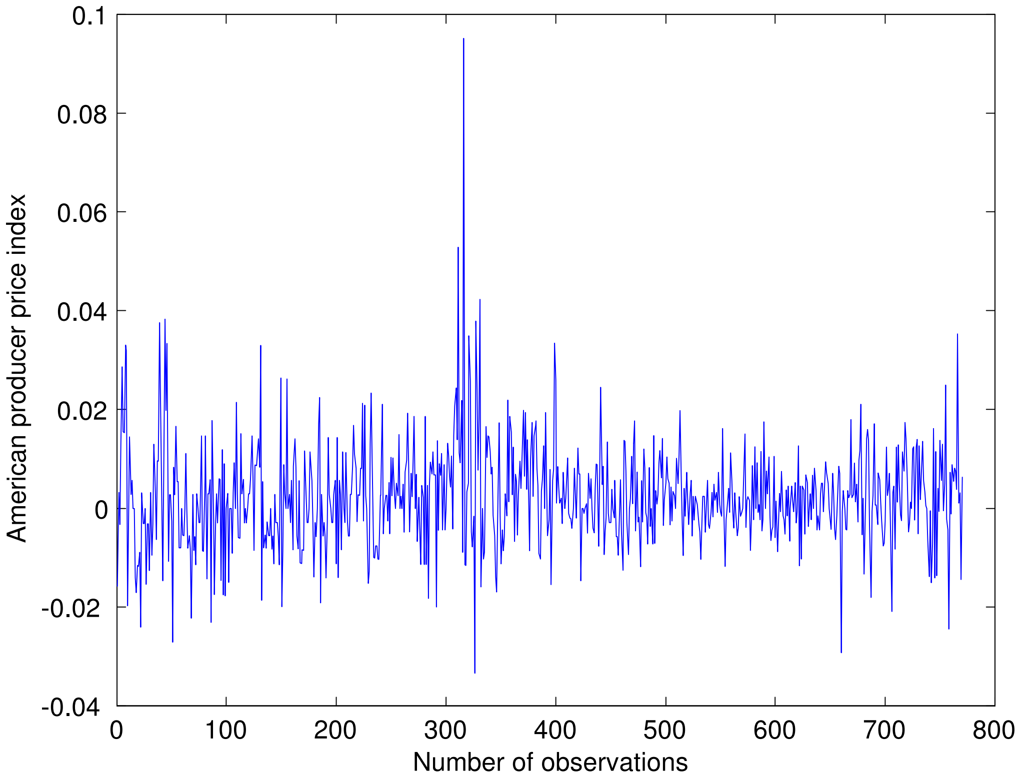

Figure 1: Monthly growth rate of the American producer price index.

[3]The gradients $\partial H\left(D_t, x_t, \hat{\Psi}\right)/\partial\Psi$ are presented in Appendix 2.

4.2 Empirical Specification

4.2.1 Nonlinearity Test

Since these models are based on autoregressive structures, the first problem we face in searching for the appropriate econometric specification is to select the right lag structure. Then, we fit an $ARFI(p)$ model assuming that the selected lag order p is the same in both regimes of the nonlinear model. We choose an autoregressive order equal to 9 from a set of candidate values ranging from 1 to 10.

The next step consists in testing whether a nonlinear model will be appropriate for this series, i.e., testing linearity against $FIBSTAR$. Table 1 displays the results of the linearity test. Using an LM test, the null hypothesis of linearity is actually rejected for all transition variables (y_{t-z}) from delay 1 to 9 except for $d = 3$. As a practical approach, we choose the delay parameter in order to minimize the p-value. The results indicate that $d = 4$ is the appropriate choice for the delay of the transition variable.

Table 1: LM-type test of nonlinearity

Transition variables	p-values
y_{t-1}	1.581×10^{-9}
y_{t-2}	1.5731×10^{-4}
y_{t-3}	0.1626
y_{t-4}	1.1422×10^{-9}
y_{t-5}	5.6677×10^{-7}
y_{t-6}	0.0016
y_{t-7}	0.0060
y_{t-8}	4.8989×10^{-4}
y_{t-9}	0.0401

4.2.2 Estimation

After having rejected a linear model against a nonlinear $FI - BSTAR$ model using an LM-type test, we proceed now with estimating the long memory $BSTAR$ model using a maximum likelihood method. The estimation results are reported in Table 2.

Table 2 shows that most of the coefficients are statistically significant in the linear and nonlinear part of the long memory $BSTAR$ model. The estimated fractional integration parameter $\hat{d}$ is equal to 0.1949 and significant at the 5% level. This indicates a strong evidence of long memory. Furthermore, all transition function parameters are significant except for the first slope parameter. Moreover, the transition is smooth ($\hat{\gamma}_2 = 0.6128$) around the neighborhood of the upper location parameter c_2.

Table 2: Summary of estimated FI-BSTAR model

Parameters	Estimates	t-statistic
d	0.1949*	(2.462)
c_1	-0.0247*	(-2.215)
γ_1	0.0763	(1.0102)
c_2	0.0213**	(9.8824)
γ_2	0.6128*	(2.3924)
ϕ_0	-0.0169*	(-2.533)
ϕ_1	-0.2313	(-0.817)
ϕ_2	0.7878**	(4.2109)
ϕ_3	-0.6938**	(-4.420)
ϕ_4	-0.1215**	(-4.197)
ϕ_5	1.4165**	(10.6749)
ϕ_6	0.9062**	(9.0448)
ϕ_7	0.1893	(1.071)
ϕ_8	0.0461	(0.5619)
ϕ_9	0.3297**	(3.0102)
θ_0	0.0327*	(2.2385)
θ_1	0.1706	(0.3194)
θ_2	-1.4435**	(-5.8001)
θ_3	1.1064**	(4.9826)
θ_4	-0.0778	(-0.6145)
θ_5	-2.5786**	(-33.7635)
θ_6	-1.5727**	(-5.5029)
θ_7	-0.3144	(-0.9038)
θ_8	-0.0118	(-0.0787)
θ_9	-0.4564**	(-2.6641)

Note: **,* indicate respectively that the coefficient is significant at the 1% and 5%, levels.

4.2.3 Diagnostic

The diagnostic of the estimated model is based on the properties of the obtained residuals. Three different tests are used to this aim: Lilliefors normality test[4], the residuals autocorrelation test as described above and a test for an $ARCH$ effect.

Table 3 presents the different diagnostic results for our $FI-BSTAR$ model. Lilliefors test statistics shows that we cannot reject the normality hypothesis at 5%. The residuals autocorrelation test based on LM statistics for long memory $BSTAR$ model provides strong evidence for no residuals autocorrelation. Additionally, we elaborate an $ARCH$ test for the autoregressive conditional heteroskedasticity in the residuals. As the no-$ARCH$ hypothesis is not rejected, this leads

[4]The Lilliefors test of normality is used because it is more powerful than other procedures for a wide range of abnormal conditions (see Abdi and Molin 2007).

us to assume a constant conditional variance in error processes.

Table 3: Misspecification tests for estimated FI-BSTAR model

Tests	$p - values$
$Lilliefors\ test$	0.162
$LM_{SC}(1)$	0.436
$LM_{SC}(4)$	0.561
$LM_{SC}(8)$	0.383
$ARCH(1)$	0.183
$ARCH(2)$	0.242
$ARCH(8)$	0.764

Note: The Lilliefors is the normality test of the residuals. $LM_{SC}(q)$ denotes the LM test of no serial correlation in residuals up to order q and $ARCH(q)$ is the LM test of no autoregressive conditional heteroscedasticity up to order q.

5 Conclusion

In this paper, we introduced the fractionally integrated Bi-parameter smooth transition model ($FI-BSTAR$ model). The $FI-BSTAR$ model allowed for regime switching based on the bi-parameter transition function and long memory behaviours. We have used the specific to general procedure to empirically specify the fractionally integrated $BSTAR$ model. As an empirical application, the $FI-BSTAR$ model is fitted to the growth rate of the American producer price index time series and the obtained results corroborate a strong evidence of this type of nonlinearity.

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Appendix 1

Let's recall that the $FI - BSTAR(p)$ model is defined as:

$$(1-L)^d y_t = x_t$$

with

$$x_t = \phi_0 + \sum_{i=1}^{p} \phi_i x_{t-i} + \left(\theta_0 + \sum_{i=1}^{p} \theta_i x_{t-i}\right) \times F(\gamma_1, \gamma_2, c_1, c_2; y_{t-z}) + \varepsilon_t$$

where $\varepsilon_t \sim NID\left(0, \sigma^2\right)$,

$$\begin{aligned} F(\gamma_1, \gamma_2, c_1, c_2; y_{t-z}) &= \frac{\exp\left[-\frac{\gamma_1 (y_{t-z}-c_1)}{\sigma_F}\right] + \exp\left[\frac{\gamma_2 (y_{t-z}-c_2)}{\sigma_F}\right]}{1 + \exp\left[-\frac{\gamma_1 (y_{t-z}-c_1)}{\sigma_F}\right] + \exp\left[\frac{\gamma_2 (y_{t-z}-c_2)}{\sigma_F}\right]} \\ \gamma_1 &> 0,\ \gamma_2 > 0,\ c_1 < c_2 \end{aligned}$$

All parameters of the $FI-BSTAR$ are presented above except σ_F which is the sample deviation of the transition variables suggested by Teräsvirta (1994) to standardize the transition variable.

The partial derivatives of log-likelihood function with respect to the $FI-BSTAR$ model are:

$$
\begin{aligned}
\frac{\partial \ln l_t}{\partial \phi} &= \frac{1}{\sigma^2} e_t w_t \\
\frac{\partial \ln l_t}{\partial \theta} &= \frac{1}{\sigma^2} e_t w_t F\left(y_{t-z}\right) \\
\frac{\partial \ln l_t}{\partial \gamma_1} &= -\frac{1}{\sigma^2} w_t(\theta' x_t) \frac{\left(y_{t-z}-c_1\right)}{\sigma_F} \exp\left(\frac{-\gamma_1\left(y_{t-z}-c_1\right)}{\sigma_F}\right)\left[1-F\left(y_{t-z}\right)\right]^2 \\
\frac{\partial \ln l_t}{\partial c_1} &= \frac{1}{\sigma^2} w_t(\theta' x_t) \frac{\gamma_1}{\sigma_F} \exp\left(\frac{-\gamma_1\left(y_{t-z}-c_1\right)}{\sigma_F}\right)\left[1-F\left(y_{t-z}\right)\right]^2 \\
\frac{\partial \ln l_t}{\partial \gamma_2} &= \frac{1}{\sigma^2} w_t(\theta' x_t) \frac{\left(y_{t-z}-c_2\right)}{\sigma_F} \exp\left(\frac{\gamma_2\left(y_{t-z}-c_2\right)}{\sigma_F}\right)\left[1-F\left(y_{t-z}\right)\right]^2 \\
\frac{\partial \ln l_t}{\partial c_2} &= -\frac{1}{\sigma^2} w_t(\theta' x_t) \frac{\gamma_2}{\sigma_F} \exp\left(\frac{\gamma_2\left(y_{t-z}-c_2\right)}{\sigma_F}\right)\left[1-F\left(y_{t-z}\right)\right]^2 \\
\frac{\partial \ln l_t}{\partial d} &= -\frac{1}{\sigma^2} w_t \sum_{j=1}^{t-1} \frac{\hat{\varepsilon}_{t-j}}{j}.
\end{aligned}
$$

Appendix 2

The needed gradients for equation (1) and (2) with respect to Ψ vector parameters are:

$$
\begin{aligned}
\frac{\partial H}{\partial \phi} &= w_t \\
\frac{\partial H}{\partial \theta} &= w_t F\left(y_{t-z}\right) \\
\frac{\partial H}{\partial \gamma_1} &= -(\theta' w_t) \frac{\left(y_{t-z}-c_1\right)}{\sigma_F} \exp\left(\frac{-\gamma_1\left(y_{t-z}-c_1\right)}{\sigma_F}\right)\left[1-F\left(y_{t-z}\right)\right]^2 \\
\frac{\partial H}{\partial \gamma_2} &= (\theta' w_t) \frac{\left(y_{t-z}-c_2\right)}{\sigma_F} \exp\left(\frac{\gamma_2\left(y_{t-z}-c_2\right)}{\sigma_F}\right)\left[1-F\left(y_{t-z}\right)\right]^2 \\
\frac{\partial H}{\partial c_1} &= (\theta' w_t) \frac{\gamma_1}{\sigma_F} \exp\left(\frac{-\gamma_1\left(y_{t-z}-c_1\right)}{\sigma_F}\right)\left[1-F\left(y_{t-z}\right)\right]^2 \\
\frac{\partial H}{\partial c_2} &= -(\theta' w_t) \frac{\gamma_2}{\sigma_F} \exp\left(\frac{\gamma_2\left(y_{t-z}-c_2\right)}{\sigma_F}\right)\left[1-F\left(y_{t-z}\right)\right]^2 \\
\frac{\partial H}{\partial d} &= -\sum_{j=1}^{t-1} \frac{\hat{\varepsilon}_{t-j}}{j}.
\end{aligned}
$$