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Convex Approximation of Bounded Rational Equilibria

Yusuke Miyazaki
SUR Co., Ltd

Hiromi Azuma
Accenture Japan Ltd

Abstract

In this paper, we consider the existence of a sequence of convex sets that has an approximation property for the equilibrium sets in the bounded rational environments. We show that the bounded rational equilibrium multivalued map is approximated with arbitrary precision in the abstract framework, a parameterized class of "general games" together with an associated abstract rationality function that is established by Anderlini and Canning (2001). As an application, we show that the existence of a selection for some bounded rational equilibria on a discontinuous region P when P is a perfect set.

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Contact: Yusuke Miyazaki - em064207@gmail.com, Hiromi Azuma - hiromiazuma@hotmail.com.
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1. Introduction

Although the postulate of unbounded rationality has dominated economic modeling for several decades, recent experimental studies (see Roth (1995)) show that people often fail to conform to some of the basic assumptions of rational decision theory so that human behavior which is based on the perfect rational choice is unrealistic.

Herbert Simon extensively investigated the issue. Even though much of his work was conceptual rather than theoretical, many economists recognized the importance of bounded rationality. However, the lack of a formal approach impeded its progress.

By defining an abstract model with bounded rationality, Anderlini and Canning (2001) showed that the model M is robust to ϵ-equilibria if and only if it is structurally stable. Their model has an advantage since it captures many common approaches to modeling bounded rationality. However, we know that equilibrium multivalued maps for some typical economic models might lose lower semicontinuity. What kind of shape are bounded rational equilibria on discontinuous points ?

Following Anderlini and Canning (2001) and Yu et al. (2009), we prove that the bounded rational equilibrium map is, under some assumptions, convex approximable. With the celebrated Michael's selection theorem, we also show a selection theorem for bounded rational equilibria on a perfect set P (which is a closed set with no isolated points) as its corollary and the essential part of this theorem is naturally to apply the result of Mazurkiewicz (1932) to show the existence of a convergent subsequence. Thus, we have a selection on a perfect set when it is contained in the set of discontinuous points. It is well known that the continuous selection theorem has some applications to differential inclusions.

The rest of the paper is organized as follows. In the next section, the model is described completely. The main result is obtained in Section 3.

2. The Model

2.1. **Notation and Assumptions.** Let (Z, d) be a metric space. We shall consider the following subsets of 2^Z :

$$\mathcal{C}_b(Z) = \{A \in 2^Z \mid A \text{ is convex bounded}\},$$

By $S(z, \sigma)$ we denote the open ball in Z with center at z and radius $\sigma > 0$. In a normed space, for notational convenience we set $S = S(0, 1)$.Given a point $a \in Z$ and a nonempty closed set $B \subseteq Z$ we define

$$r(a, B) = \inf\{d(a, b) \mid b \in B\}$$

If A and B are nonempty closed subsets of Z, we define

$$h^\star(A, B) = \sup\{r(a, B) \mid a \in A\}$$

$$h(A, B) = \sup\{h^\star(A, B), h^\star(B, A)\}.$$

A multivalued map $G : Y \to 2^Z$ is **convex (closed) valued** if $G(y)$ is convex (closed) for all $y \in Y$.

Definition 2.1. *A multivalued map $F : Y \to 2^Z$ is called $h^\star$-**u.s.c.** if for every $y_0 \in Y$ and for every $\epsilon > 0$ there is a $\delta > 0$ such that $h^\star(F(y), F(y_0)) < \epsilon$ for every $y \in S(y_0, \delta)$.*

Definition 2.2. *A multivalued map $F : Y \to 2^Z$ is called $h^\star$-**l.s.c.** if for every $y_0 \in Y$ and for every $\epsilon > 0$ there is a $\delta > 0$ such that $h^\star(F(y_0), F(y)) < \epsilon$ for every $y \in S(y_0, \delta)$.*

Definition 2.3. *A multivalued map $F : Y \to 2^Z$ which is both u.s.c. (resp. $h^\star$-u.s.c.) and l.s.c. (resp. $h^\star$-l.s.c.) is called **continuous** (resp. h-**continuous**).*

Definition 2.4. *A real valued function $f : Y \to \mathbb{R}$ defined on a convex set Y in a vector space is called **convex** if for any two points y_1 and y_2 in Y and any $t \in [0, 1]$, $f(ty_1 + (1-t)y_2) \le tf(y_1) + (1-t)f(y_2)$. A multivalued map is **convex** if its graph is convex.*

Definition 2.5. *Assume that Y is a metric space, while Z denotes a (real) normed space. A multivalued map $\phi : Y \to \mathcal{C}_b(Z)$ is called **convex approximable** if there is a sequence $\{G_n\}_{n\in\mathbb{N}}$ of closed valued h-continuous multivalued maps $G_n : Y \to \mathcal{C}_b(Z)$ with the following properties :*

(i) *for every $n \in \mathbb{N}$ and $y \in Y$ there is a $\sigma_n(y) > 0$ such that $\phi(S(y, \sigma_n(y))) \subseteq G_n(y)$*
(ii) *$G_n(y)$ is a decreasing sequence for any $y \in Y$.*
(iii) *$h($ graph G_n, graph $\phi) \to 0$ as $n \to \infty$.*

Definition 2.6. *Let $F : Y \to 2^Z$ be a multivalued map and $f : Y \to Z$ be a singlevalued map. f is called a **selection** of F if $f(y) \in F(y)$ for all $y \in Y$.*

2.2. **The Model.** Consider an abstract model $M = \{\Lambda, X, \{F_\lambda\}_{\lambda\in\Lambda}, R\}$ where

(i) Λ is a nonempty parameter space
(ii) X is an action space
(iii) $\{F_\lambda\}_{\lambda\in\Lambda}$ is a parameterized family of multivalued maps, and each multivalued map F_λ from X to X which defines a natural multivalued map

$$f : \Lambda \to 2^X$$

such that it assigns $\lambda \in \Lambda$ to the fixed point set of F_λ, $Fix(F_\lambda) = \{x \in X | x \in F_\lambda(x)\}$.
(iv) $R : Graph(f) \to \mathbb{R}_+ = \{r \in \mathbb{R} \mid r \ge 0\}$ is a rationality function with $R(\lambda, x) = 0$ corresponding to the full rationality.

For any $\lambda \in \Lambda$ and any $\epsilon \ge 0$, the set of ϵ -equilibria at λ is defined as

$$E(\lambda, \epsilon) = \{x \in f(\lambda) \mid R(\lambda, x) \le \epsilon\}.$$

Particularly, the set of equilibria at λ is defined as

$$E(\lambda) = E(\lambda, 0).$$

We regard the set of ϵ -equilibria at λ as **the bounded rational equilibrium multivalued map** $E : \Lambda \times \mathbb{R}_+ \to 2^X$ and also the set of equilibria at λ as **the equilibrium multivalued map** $E_0 : \Lambda \to 2^X$.

We pose the following assumptions(see Yu et al. (2009)).

Assumption 2.7. *Λ and X are Banach spaces.*

Assumption 2.8. *f is an upper semicontinuous multivalued map, and for any $\lambda \in \Lambda$, $Fix(F_\lambda)$ is nonempty and compact.*

Assumption 2.9. *R is a lower semicontinuous function.*

Assumption 2.10. *$E(\lambda) \neq \emptyset$ for any $\lambda \in \Lambda$.*

3. Main Result

Theorem 3.1. *If the rationality function R is convex, F_λ is a convex multivalued map and $E(\lambda)$ is bounded, then the equilibrium multivalued map E_0 is convex approximable.*

Proof. Note that the equilibrium multivalued map E_0 is upper semicontinuous(see Theorem 2.1 of Yu et al. (2009)). By the convexity of R and F_λ, the equilibrium set $E(\lambda)$ is convex. Thus, bounded rational equilibria of M is convex approximable by Theorem 2.3 of DeBlasi and Myjak (1986). □

In many economic models, parameter spaces and action spaces are subsets of Euclidean space. The bounded rational equilibrium multivalued map is upper semicontinuous in this environment.

Lemma 3.2. *When $\Lambda \subseteq \mathbb{R}^p$ and $X \subseteq \mathbb{R}^q$, then ϵ-equilibrium multivalued map is upper semicontinuous for any $\epsilon \geq 0$.*

Proof. Note that $E(\lambda, \epsilon) = Fix(F_\lambda) \cap \{x \in X \mid R(\lambda, x) \leq \epsilon, (\lambda, x) \in Graph(f)\}$. Set $W : \Lambda \times \mathbb{R}_{++} \to 2^X$ such that $W(\lambda, \epsilon) = \{x \in X \mid R(\lambda, x) \leq \epsilon, (\lambda, x) \in Graph(f)\}$. Note that $Fix(F_\lambda)$ is a compact set and f is upper semicontinuous at λ.

$$\begin{aligned} Graph(W) &= \{(\lambda, \epsilon, x) \in \Lambda \times \mathbb{R}_{++} \times X \mid x \in W(\lambda, \epsilon), (\lambda, x) \in Graph(f)\} \\ &= \{(\lambda, \epsilon, x) \in \Lambda \times \mathbb{R}_{++} \times X \mid R(\lambda, x) \leq \epsilon, (\lambda, x) \in Graph(f)\} \end{aligned}$$

is closed because defining $\tilde{R}(\lambda, \epsilon, x) = R(\lambda, x) - \epsilon$, we can easily prove the lower semicontinuity of $\tilde{R}(\lambda, \epsilon, x)$ at (λ, ϵ, x). Given arbitrary small number $c > 0$, since R is lower semicontinuous at (λ, x), there is a neighborhood V of (λ, x) corresponding to $\frac{c}{2}$ such that for any $(\lambda', x') \in V$,

$$R(\lambda', x') \geq R(\lambda, x) - \frac{c}{2} > R(\lambda, x) - c.$$

Since V is a subset of $\mathbb{R}^{p+q}$, V includes a $(p+q)$ - dimensional open hypercube

$$HC_V^c = \{y \in \mathbb{R}^{p+q} \mid \|y - (\lambda, x)\|_\infty < c\}$$

for sufficiently small $c \in \mathbb{R}_{++}$ and that is of the form $\prod_{i=1}^{p+q}(x_i, y_i)$ for $x_i < y_i$. We choose the neighborhood $U \subseteq \mathbb{R}^p \times \mathbb{R}_+ \times \mathbb{R}^q$ of (λ, ϵ, x) such that

$$U = \{\tilde{\lambda} \in \mathbb{R}^p \mid (\tilde{\lambda}, x) \in HC_V^c\} \times \{\tilde{\epsilon} \in \mathbb{R}_+ \mid |\tilde{\epsilon} - \epsilon| < \frac{c}{2}\} \times \{\tilde{x} \in \mathbb{R}^q \mid (\lambda, \tilde{x}) \in HC_V^c\}.$$

For $(\hat{\lambda}, \hat{\epsilon}, \hat{x}) \in U$, we have the inequality

$$R(\hat{\lambda}, \hat{x}) \geq R(\lambda, x) + (\hat{\epsilon} - \epsilon) - c,$$

(if $\hat{\epsilon} - \epsilon < \frac{c}{2}$) which leads to

$$\tilde{R}(\hat{\lambda}, \hat{\epsilon}, \hat{x}) \geq \tilde{R}(\lambda, \epsilon, x) - c.$$

(If $\epsilon - \hat{\epsilon} < \frac{c}{2}$,

$$R(\hat{\lambda}, \hat{x}) \geq R(\lambda, x) + (\epsilon - \hat{\epsilon}) - c$$

holds.) This establishes the lower semicontinuity of $\tilde{R}$ at (λ, ϵ, x). Therefore, the graph of W is a closed set, so the ϵ-equilibrium multivalued map E is upper semicontinuous at (λ, ϵ) by Proposition 1.4.9 of Aubin and Frankowska (1990). Because (λ, ϵ) is arbitrary, E is upper semicontinuous. □

Theorem 3.3. *Assume that Λ and X are subsets of finite dimensional Euclidean space. If the rationality function R is convex, F_λ is a convex multivalued map and $E(\lambda, \epsilon)$ is bounded, then the bounded rational equilibrium multivalued map E is convex approximable.*

Proof. By the convexity of R and F_λ, the equilibrium set $E(\lambda, \epsilon)$ is convex. Thus, bounded rational equilibria of M is convex approximable by Theorem 2.3 of DeBlasi and Myjak (1986). □

For almost all parameters, the equilibrium multivalued map is continuous(see Yu et al. (2009), Yu and Yu (2007), Yu (1999)) i.e. there possibly exists a discontinuous point such that the equilibrium at that point is hard to be approximable. However, under some conditions, a convex equilibrium set $E(\lambda, \epsilon)$ can be approximated by a sequence of a class of multivalued maps even if the equilibrium multivalued map E is discontinuous at $(\lambda, \epsilon) \in \Lambda \times \mathbb{R}_+$.

3.1. **Application : Selection Theorem.** The equilibrium multivalued map is continuous except for some points. At these discontinuous points, it surely can be difficult to analyze the bounded rational equilibria. However, the above approximation result yields the existence of a selection for some bounded rational equilibria on a discontinuous region P when P is a perfect set.

In this section, we assume that Λ and X are compact subsets of finite dimensional Euclidean spaces and that the rationality function R is continuous.

Theorem 3.4 (Selection Theorem for Bounded Rational Equilibria)**.** *Denote the set of discontinuous points by $\mathcal{D}$, i.e.*

$$\mathcal{D} = \{(\lambda, x) \in \Lambda \times \mathbb{R}_+ \mid E \text{ is discontinuous at } (\lambda, x)\}.$$

Assuming that for any $(\lambda, \epsilon) \in \mathcal{D}$, the restriction of the equilibrium multivalued map E to the set $\mathcal{D}$ is bounded. If there is a perfect subset $P \subseteq \mathcal{D}$, then there is a perfect set $P^* \subseteq P$ such that we have a continuous selection of E on P^* under the assumptions of Theorem 3.3.

Proof. Denote the restriction of the equilibrium multivalued map E to the set $\mathcal{D}$ by $E_{\mathcal{D}} : \mathcal{D} \to 2^X$. Then, $E_{\mathcal{D}}$ is convex approximable and for a sequence of multivalued maps $\{G_n\}_{n \in \mathbb{N}}$, we have a sequence $\{g_n\}_{n \in \mathbb{N}}$ of continuous selections by Michael selection theorem, i.e. for any $n \in \mathbb{N}$ and $(\lambda, \epsilon) \in \mathcal{D}$, $g_n(\lambda, \epsilon) \in G_n(\lambda, \epsilon)$. Therefore, for any $(\lambda, \epsilon) \in \mathcal{D}$, $g_n(\lambda, \epsilon) \in G_1(\lambda, \epsilon)$ for every $n \in \mathbb{N}$. Since G_1 is bounded, h-continuous and compact-valued, $\{g_n\}_{n \in \mathbb{N}}$ is uniformly bounded on $\mathcal{D}$.

Thus, there is a perfect set $P^* \subseteq P$ such that $\{g_n\}_{n\in\mathbb{N}}$ has a convergent subsequence $\{\tilde{g}_n\}_{n\in\mathbb{N}}$ on P^* by Mazurkiewicz (1932). Denote the limit by $g = \lim_{n\to\infty} \tilde{g}_n$. Note that $\lim_{n\to\infty} h(G_n(\lambda,\epsilon), E(\lambda,\epsilon)) = 0$ by Theorem 4.2 of DeBlasi and Myjak (1986). For any $(\lambda,\epsilon) \in \mathcal{D}$,

$$h(g(\lambda,\epsilon), E(\lambda,\epsilon)) \leq h(g(\lambda,\epsilon), g_n(\lambda,\epsilon)) + h(g_n(\lambda,\epsilon), G_n(\lambda,\epsilon)) + h(G_n(\lambda,\epsilon), E(\lambda,\epsilon)).$$

Then $h(g(\lambda,\epsilon), E(\lambda,\epsilon)) = 0$. Since $E(\lambda,\epsilon)$ is a closed subset of X, $g(\lambda,\epsilon) \in E(\lambda,\epsilon)$, i.e. g is a selection of E. The proof is completed. □

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