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# Plurality with run-off and triangulars

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## Abstract

In this paper we explore some aspects of the plurality rule with run-off, with respect to triangulars. Over the universal domain and under the impartial anonymous culture (IAC), we provide frequencies that (i) a triangular occurs, and (ii) a Condorcet winner - or a Condorcet loser - wins the election after a triangular. We also evaluate the likelihood of a triangular over the restricted domain of single-peaked preferences.

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# 1 Introduction

Under the plurality rule with run-off, the second round - if necessary - is usually organized between the two candidates with the most first-place votes. However, in some versions of this voting system, all candidates over a given percentage of votes are allowed to compete for the run-off. For example, in french cantonal elections, all candidates with at least 12.5% of first-place votes are potential candidates for the run-off. In such elections, a triangular refers to voting situations at which no candidate gets an absolute majority of votes after the first round and only three of them collect enough votes, according to the chosen threshold $0 < \theta \leq 1$.

Plurality with run-off has been studied by several authors. For example, in the recent literature, Martinelli (2002) and O'Neill (2006) provide many results on this topic. Martinelli (2002) compares two voting methods commonly used in presidential elections: simple plurality voting and plurality with run-off. In a model with three candidates, the link between information aggregation and the need for coordination is underlined. It is shown that plurality with run-off leads to higher expected utility for the majority than simple plurality voting if the information held by voters about the candidates is not very accurate. O'Neill (2006) investigates when a run-off election is desirable and when a plurality result is good enough. A run-off election increases the likelihood that the Condorcet winner will be elected but also entails additional costs. The models allow governments to make more informed choices in creating rules to decide when to hold run-off elections.

Although each of the above mentioned papers studies simple plurality, plurality with run-off and configurations of preferences with a Condorcet winner, the focus of our paper is quite different. Our contribution in this paper consists in the evaluation of theoretical frequencies of triangulars under the universal domain and also under the single-peakedness - intuitively, single-peaked preferences are those preferences that are compatible with a left-right ideological axis - domain. Evaluations are based on the impartial anonymous culture (IAC) hypothesis as distinguished from the impartial culture or maximal culture. Our choice of the IAC assumption is partially justified by the fact that we deal with anonymous voters.

The paper is organized as follows: in section 2 is dedicated to presenting triangulars opportunities, while in section 3 we evaluate occurrences of triangulars before concluding the paper in section 4.

# 2 Triangulars possibilities

Consider an election in which $A = \{a_1, ..., a_m\}$ is a finite set of candidates, and $N$ is the set of $n$ voters, whose preferences are aggregated in order to determine the elected candidate. Every voter reports a linear order (complete, transitive and antisymmetric binary relation) over the set $A$ of candidates, that is, with three candidates, one of the six following preference relations: $R_1 : a_1a_2a_3$; $R_2 : a_1a_3a_2$; $R_3 : a_2a_1a_3$; $R_4 : a_2a_3a_1$; $R_5 : a_3a_1a_2$; $R_6 : a_3a_2a_1$.

Let $n_j$ denote the number of voters whose preference relation is $R_j$ ($j = 1, 2, ..., 6$). We must then have $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n$. A anonymous profile is a vector $s = (n_1, n_2, n_3, n_4, n_5, n_6)$.

In this section, we give the set of inequalities that characterize anonymous profiles at which a triangular occurs. We shall successively consider the following cases : (i) universal domain (all individual preferences are susceptible to be reported), (ii) single-peaked preferences, (iii) configurations of preferences selecting a Condorcet Winner - a candidate that beats every other candidate in pairwise majority contests, and (iv) configurations of preferences selecting a Condorcet loser - a candidate that is beaten by every other candidate in pairwise majority contests. Note that in some cases, there may be ties; they will be broken by choosing the alternative with the greatest index.

**Example 1** *Consider the anonymous profile below, with a run-off threshold $\theta = 12.5\%$:*

| *anonymous profile 1* | | | | | | *anonymous profile 2* | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| *number of voters* | | | | | | *number of voters* | | | | | |
| *33* | *0* | *25* | *0* | *0* | *42* | *33* | *12* | *6* | *3* | *6* | *40* |
| $a_1$ | $a_1$ | $a_2$ | $a_2$ | $a_3$ | $a_3$ | $a_1$ | $a_1$ | $a_2$ | $a_2$ | $a_3$ | $a_3$ |
| $a_2$ | $a_3$ | $a_1$ | $a_3$ | $a_1$ | $a_2$ | $a_2$ | $a_3$ | $a_1$ | $a_3$ | $a_1$ | $a_2$ |
| $a_3$ | $a_2$ | $a_3$ | $a_1$ | $a_2$ | $a_1$ | $a_3$ | $a_2$ | $a_3$ | $a_1$ | $a_2$ | $a_1$ |

*In anonymous profile 1, all three candidates are qualified for the second round, while in anonymous profile 2, only $a_1$ and $a_3$ are allowed to pursue the competition.*

**Proposition 2** *With 3 candidates, universal domain and threshold $\theta$, a triangular arises under plurality with run-off iff:*

$$\begin{cases} \theta n \leq n_1 + n_2 \leq \frac{n}{2} \\ \theta n \leq n_3 + n_4 \leq \frac{n}{2} \\ \theta n \leq n_5 + n_6 \leq \frac{n}{2} \\ \displaystyle\sum_{i=1}^{6} n_i = n \end{cases}$$

Now, the reader can easily construct an example illustrating the election of a Condorcet Winner in a triangular.

The proposition below summarizes all such situations.

**Proposition 3** *Given a threshold $\theta$, Condorcet winner is elected after a triangular iff :*

$$\left\{\begin{array}{c}\theta n \le n_1+n_2 \le \frac{n}{2}\\ \theta n \le n_3+n_4 \le \frac{n}{2}\\ \theta n \le n_5+n_6 \le \frac{n}{2}\\ n_1+n_2+n_5 > \frac{n}{2}\\ n_1+n_2+n_3 > \frac{n}{2}\\ n_1+n_2 > n_3+n_4\\ n_1+n_2 > n_5+n_6\\ \sum_{i=1}^{6} n_i = n\end{array}\right. \text{ or } \left\{\begin{array}{c}\theta n \le n_1+n_2 \le \frac{n}{2}\\ \theta n \le n_3+n_4 \le \frac{n}{2}\\ \theta n \le n_5+n_6 \le \frac{n}{2}\\ n_3+n_4+n_6 \ge \frac{n}{2}\\ n_1+n_3+n_4 > \frac{n}{2}\\ n_1+n_2 \le n_3+n_4\\ n_5+n_6 < n_3+n_4\\ \sum_{i=1}^{6} n_i = n\end{array}\right. \text{ or } \left\{\begin{array}{c}\theta n \le n_1+n_2 \le \frac{n}{2}\\ \theta n \le n_3+n_4 \le \frac{n}{2}\\ \theta n \le n_5+n_6 \le \frac{n}{2}\\ n_4+n_5+n_6 \ge \frac{n}{2}\\ n_2+n_5+n_6 \ge \frac{n}{2}\\ n_1+n_2 \le n_5+n_6\\ n_3+n_4 \le n_5+n_6\\ \sum_{i=1}^{6} n_i = n\end{array}\right.$$

Similarly, the proposition below summarizes all situations at which a Condorcet loser is elected after a triangular.

**Proposition 4** *Given a threshold $\theta$, a Condorcet loser is elected after a triangular contest iff :*

$$\left\{\begin{array}{c}\theta n \le n_1+n_2 \le \frac{n}{2}\\ \theta n \le n_3+n_4 \le \frac{n}{2}\\ \theta n \le n_5+n_6 \le \frac{n}{2}\\ n_3+n_4+n_6 \ge \frac{n}{2}\\ n_4+n_5+n_6 \ge \frac{n}{2}\\ n_1+n_2 > n_3+n_4\\ n_1+n_2 > n_5+n_6\\ \sum_{i=1}^{6} n_i = n\end{array}\right. \text{ or } \left\{\begin{array}{c}\theta n \le n_1+n_2 \le \frac{n}{2}\\ \theta n \le n_3+n_4 \le \frac{n}{2}\\ \theta n \le n_5+n_6 \le \frac{n}{2}\\ n_1+n_2+n_5 > \frac{n}{2}\\ n_2+n_5+n_6 \ge \frac{n}{2}\\ n_1+n_2 \le n_3+n_4\\ n_5+n_6 < n_3+n_4\\ \sum_{i=1}^{6} n_i = n\end{array}\right. \text{ or } \left\{\begin{array}{c}\theta n \le n_1+n_2 \le \frac{n}{2}\\ \theta n \le n_3+n_4 \le \frac{n}{2}\\ \theta n \le n_5+n_6 \le \frac{n}{2}\\ n_1+n_2+n_3 > \frac{n}{2}\\ n_1+n_3+n_4 > \frac{n}{2}\\ n_1+n_2 \le n_5+n_6\\ n_3+n_4 \le n_5+n_6\\ \sum_{i=1}^{6} n_i = n\end{array}\right.$$

We now consider situations in which all individuals report single-peaked preferences. In fact with three alternatives, we suppose that $a_1a_2a_3$ is the ideological ranking. As consequence, only $P_1 : a_1a_2a_3$, $P_2 : a_2a_1a_3$, $P_3 : a_2a_3a_1$ and $P_4 : a_3a_2a_1$ can be reported. In the same way, with $a_1a_2a_3a_4$ as the ideological ranking, the eight ranking that can be observed are the followings $L_1 : a_1a_2a_3a_4$, $L_2 : a_2a_1a_3a_4$, $L_3 : a_2a_3a_1a_4$, $L_4 : a_2a_3a_4a_1$, $L_5 : a_3a_2a_1a_4$, $L_6 : a_3a_2a_4a_1$, $L_7 : a_3a_4a_2a_1$ and $L_8 : a_4a_3a_2a_1$. The next two propositions describe all situations at which a triangular occurs, with three and four alternatives, respectively.

**Proposition 5** *Given a threshold* $\theta$*, a triangular occurs under the single-peakedness domain, with* 3 *candidates iff:*

$$\begin{cases} \theta n \leq n_1 \leq \frac{n}{2} \\ \theta n \leq n_2 + n_3 \leq \frac{n}{2} \\ \theta n \leq n_4 \leq \frac{n}{2} \\ \sum_{i=1}^{4} n_i = n \end{cases}$$

**Proposition 6** *Given a threshold* $\theta$*, a triangular occurs under the single-peakedness domain, with* 4 *candidates iff:*

$$\begin{cases} \theta n \leq n_1 \leq \frac{n}{2} \\ \theta n \leq n_2 + n_3 + n_4 \leq \frac{n}{2} \\ \theta n \leq n_5 + n_6 + n_7 \leq \frac{n}{2} \\ n_8 < \theta n \\ \sum_{i=1}^{8} n_i = n \end{cases} \quad or \quad \begin{cases} \theta n \leq n_1 \leq \frac{n}{2} \\ \theta n \leq n_2 + n_3 + n_4 \leq \frac{n}{2} \\ n_5 + n_6 + n_7 < \theta n \\ \theta n \leq n_8 \leq \frac{n}{2} \\ \sum_{i=1}^{8} n_i = n \end{cases}$$

$$\begin{cases} \theta n \leq n_1 \leq \frac{n}{2} \\ n_2 + n_3 + n_4 < \theta n \\ \theta n \leq n_5 + n_6 + n_7 \leq \frac{n}{2} \\ \theta n \leq n_8 \leq \frac{n}{2} \\ \sum_{i=1}^{8} n_i = n \end{cases} \quad or \quad \begin{cases} n_1 < \theta n \\ \theta n \leq n_2 + n_3 + n_4 \leq \frac{n}{2} \\ \theta n \leq n_5 + n_6 + n_7 \leq \frac{n}{2} \\ \theta n \leq n_8 \leq \frac{n}{2} \\ \sum_{i=1}^{8} n_i = n \end{cases}$$

# 3 Evaluation of triangulars occurrences

As said in the introduction of this paper, we are concerned with the quantitative evaluation of triangulars possibilities. Our calculations are based on a probabilistic assumption known under the name of impartial anonymous culture (IAC). Under IAC, voters are anonymous, in the sense that their identity does not matter: if we permute the preferences of two individuals, this will have no consequence on the outcome of the vote.

Frequencies of triangulars are calculated as:

$$\frac{\textbf{Number of anonymous profiles at which a triangular is possible}}{\textbf{Total number of anonymous profiles}}$$

The method used to compute these frequencies is based on Gehrlein and Fishburn (1976) and is the same as the one in Mbih et al. (2008). All technical details can be provided by the author upon simple request.

In all propositions below, we only provide two examples of formulae we obtain from our calculations. Given a set of three candidates, we first present the frequency $F(UD_3, n)$ of anonymous profiles at which a triangular occurs over the universal domain while $\theta = 12.5\%$. Thereafter we provide for large electorates - that is when $n$ tends to infinity - the ratio $G(UD_3, \theta)$ of anonymous profiles at which a triangular occurs over the universal domain for any $\theta \in ]0, 1]$. Table 1 in the appendix summarizes frequencies derived from all such polynomials.

**Proposition 7** *Suppose* $|A| = 3$. *Then the frequency of triangulars over the universal domain is*

$$F(UD_3, n) = \begin{cases} \frac{3191n^5+46\,770n^4+283\,040n^3+885\,120n^2+1439744n+983040}{8192(n+1)(n+2)(n+3)(n+4)(n+5)} \\ \hfill \text{if } n \equiv 0 \text{ mod } 8 \\ \frac{3191n^5+32\,925n^4+122\,630n^3+221\,850n^2+279\,539n+261\,465}{8192(n+1)(n+2)(n+3)(n+4)(n+5)} \\ \hfill \text{if } n \equiv 5 \text{ mod } 8 \end{cases}$$

**Proposition 8** *When* $n$ *tends to infinity, the frequency of triangulars over the universal domain is*

$$G(UD_3, \theta) = \begin{cases} \frac{7}{16} - 30\theta^3 + 45\theta^4 - 12\theta^5 \\ \hfill \textit{for } 0 \leq \theta \leq \frac{1}{4} \\ 1 - 30\theta^2 + 60\theta^3 + 45\theta^4 - 108\theta^5 \\ \hfill \textit{for } \frac{1}{4} \leq \theta \leq \frac{1}{3} \end{cases}$$

We obtain the following figure:

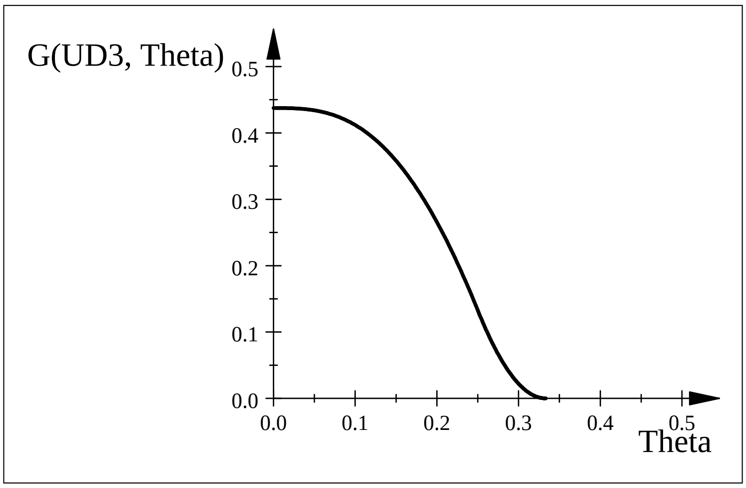


Table 1: The frequency of triangulars over the universal domain when n tends infinity

$$\theta \max = 0 \rightarrow F'(UD_3, \theta \max) = 0,4375 \text{ ; } \theta \min = \frac{1}{3} \rightarrow F'(UD_3, \theta \min) = 0$$

# 4 Concluding and remarks

This work provides information on the anonymous profiles which admit the possibility of a triangular, under plurality with run-off and some usual domain assumptions such as universal domain or single-peaked preferences. We also evaluate how often plurality with run-off may result in the election, in the second round, of a Condorcet winner or a Condorcet loser after a triangular.

With $\theta = 12,5\%$, it is shown that when $n$ tends to infinity, the plurality with run-off allows for the possibility of a triangular in universal domain ($UD_3$ [38.9%]) and restricted domains ($RD_3$ [20.3%], $RD_4$ [15.9%]) of individual preferences.

On a practical viewpoint, the use of triangulars is somewhat controversial. They seem to lead to a better representativeness and reduce the risk of electoral bargaining, but they can also lead to the election of a Condorcet loser.

On a theoretical viewpoint, it would be interesting to know how much some other rules are sensitive to the possibility of triangulars; and especially, in a comparative perspective, we could certainly have a better knowledge of some iterative positional voting rules (e.g. plurality, anti-plurality) by examining their sensitivities to triangulars.

# References


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# Appendix

**Table 1** : **Frequencies of triangulars**
**(for $\boldsymbol{\theta}$ =12, 5%)**

| $n$ | $UD_3$ | $CW_3$ | $CL_3$ | $RD_3$ | $RD_4$ |
|---|---|---|---|---|---|
| *8* | 0.403 | 0.198 | 0.0210 | 0.267 | 0.138 |
| *10* | 0.299 | 0.137 | 0.0206 | 0.182 | 0.133 |
| *18* | 0.338 | 0.158 | 0.0223 | 0.195 | 0.146 |
| *25* | 0.309 | 0.137 | 0.0225 | 0.171 | 0.135 |
| *38* | 0.381 | 0.183 | 0.0226 | 0.212 | 0.155 |
| *43* | 0.346 | 0.161 | 0.0219 | 0.188 | 0.146 |
| *53* | 0.358 | 0.168 | 0.0220 | 0.194 | 0.156 |
| *58* | 0.373 | 0.180 | 0.0226 | 0.202 | 0.156 |
| *63* | 0.367 | 0.174 | 0.0222 | 0.198 | 0.150 |
| *68* | 0.380 | 0.181 | 0.0226 | 0.205 | 0.156 |
| *73* | 0.359 | 0.170 | 0.0220 | 0.191 | 0.151 |
| *95* | 0.374 | 0.168 | 0.0222 | 0.200 | 0.153 |
| $\infty$ | 0.389 | 0.191 | 0.0224 | 0.203 | 0.159 |

In Table 1 above, the threshold chosen is $\theta = 12.5\%$ (or equivalently $\theta = 1/8$). Several types of frequency on triangulars are presented with respect to distinct domain asumptions. More precisely, the frequencies are computed over (i) $UD_3$ : the universal domain with three alternatives, (ii) $CW_3$ : the set of all profiles with three alternatives at which the Condorcet winner is elected after a triangular, (iii) $CL_3$ : the set of all profiles with three alternatives at which the Condorcet loser is elected after a triangular, (iv) $RD_3$ : the domain of single-peaked preferences with three candidates, and (v) $RD_4$ : the domain of single-peaked preferences with four candidates.