E C O N O M I C S   B U L L E T I N

Interiority of the optimal population growth rate with endogenous fertility

Gemma Abio
Universitat de Barcelona

Abstract

This paper analyzes the interiority of the optimal population growth rate in a two−period overlapping generations model with endogenous fertility as compared to the case with exogenous fertility analyzed by Samuelson (1975) and Deardorff (1976).

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1 Introduction

Samuelson (1975) analyzed the issue of the optimal growth rate for population in Diamond's (1965) classical model of two overlapping generations. Samuelson's analysis was followed by a criticism –Deardorff (1976)– concerning the interiority of the optimal solution in the planner's problem. In particular, Deardorff showed that if both the utility and the production function were of the typical Cobb-Douglas type, there would be no interior rate of population growth that maximized the utility of the representative agent at the steady state; and the same would happen for several other specifications of preferences and technology. Later, Michel and Pestieau (1993) analyzed this issue considering more general utility and production functions, of the CES type. They concluded that interiority of the optimal population growth rate required complementarity either in consumption or in production. All these papers consider the agents' fertility decision as being exogenous.

With endogenous fertility, several papers (e.g. Bental 1989, Eckstein and Wolpin 1985) have analyzed the optimal population growth rate, while the issue of its interiority has not always been the focus of those papers. Schweizer (1996) emphasizes the need to check the existence of an interior solution in such a setting, establishing an isomorphism between Samuelson's problem and the optimal provision of local public goods.

This paper investigates in what cases the optimal population growth rate is interior when fertility is endogenous –by introducing both a cost and a taste for children– in Samuelson's model. Contrary to the results obtained by Deardorff (1976) in the exogenous fertility case, we show that there is the possibility of having an interior global maximum of utility in the case of a Cobb-Douglas specification of utility and technology. Our results show that, while the taste for children does not play any role in avoiding a corner solution, the introduction of a cost of children is crucial.

2 Optimal Population Growth with Endogenous Fertility

To endogenize fertility in Samuelson's (1975) model of overlapping generations, we suppose individuals derive utility from having descendants and we assume children are costly. The number of children per individual is $n_t \equiv \frac{N_{t+1}}{N_t}$, where N_t is the size of the generation born in period t[1].

Individual preferences are represented by the following utility function:

$$U_t(n_t, c_t, d_{t+1}) = \gamma \log(n_t) + (1-\gamma)\left[\log(c_t) + \beta \log(d_{t+1})\right] \quad (1)$$

where c_t and d_{t+1} are consumption when adult and when old respectively, $\beta \in [0,1]$ is the subjective discount factor and $\gamma \in [0,1]$ is a parameter reflecting the taste for children. We assume absence of altruism, in the sense that agents do not value the utility of their children.

[1]Some authors, including Samuelson, consider $\frac{N_{t+1}}{N_t} = 1+n$, so that n is the rate of population growth. Using our notation, n is the number of children of the representative individual. This latter approach is used in most papers with endogenous fertility.

The cost of children is assumed to be a time cost z per child; this implies the endogeneity of the labor supply, because it introduces a trade-off between working in the labor market and raising children. As an adult, each individual devotes a share zn_t of their time to raising children and a share $(1 - zn_t)$ to working in the market. Assuming each agent has an endowment of 1 unit of time, with the Cobb-Douglas specification and in per-capita terms, the production function can be written as:

$$f(k_t, l_t) = Ak_t^{\alpha} l_t^{1-\alpha} \tag{2}$$

where $l_t = 1 - zn_t$ is the labor supply per individual, k_t is the stock of capital per capita, A is a technological parameter and $\alpha \in [0, 1]$ is a parameter that represents the share of income that goes to capital earnings. For simplicity, we assume capital totally depreciates in the production process.

As in Samuelson (1975), optimality is defined here as the allocation that maximizes the steady state utility of the representative individual:

$$\max_{c,d,n,k} U(n, c, d) = \gamma log(n) + (1 - \gamma) [log(c) + \beta log(d)] \tag{3}$$

subject to the resource constraint:

$$Ak^{\alpha} l^{1-\alpha} = c + \frac{d}{n} + nk \tag{4}$$

with $l = 1 - zn$.

The first order conditions (FOC) for an interior solution are:

$$\frac{d^*}{\beta c^*} = n^* \tag{5}$$

$$\frac{d^*}{(n^*)^2} + \frac{\gamma}{1-\gamma} \frac{c^*}{n^*} = k^* + zA(1 - \alpha) \left(\frac{k^*}{l^*}\right)^{\alpha} \tag{6}$$

$$A\alpha \left(\frac{l^*}{k^*}\right)^{1-\alpha} = n^* \tag{7}$$

The first equation gives the optimal intertemporal allocation of consumption, while equation (7) determines the optimal level of capital per capita, defined by the golden rule. Equation (6) determines the optimal number of children. The marginal benefit (MB) of children includes two terms: the intergenerational transfer effect[2] (ITE) and the marginal utility of children, expressed in terms of first-period consumption. The marginal cost (MC), on the other hand, is the sum of the capital dilution effect[3] (KDE) and the loss in production due to the time cost of children. With exogenous fertility, Samuelson determines the optimal population growth rate by equalizing the ITE to the KDE.

Using the FOC, we can obtain the indirect utility function, the MB and the MC of children in terms of n. After verifying that the latter are both decreasing and convex as a function of n, we obtain the difference between the two as:

$$MB - MC = \left(\frac{A\alpha}{n}\right)^{\frac{1}{1-\alpha}} \frac{\Gamma - \Theta zn}{\alpha(1 - \gamma)(1 + \beta)} \tag{8}$$

[2]This term captures the fact that, when the population grows, there are more adult working individuals to support each old.

[3]According to this effect, when the population grows, the stock of capital must be expanded in order for the same capital-labor ratio to be maintained.

where

$$\Gamma \equiv \gamma + \beta(1-\gamma) - \alpha\left[1 + 2\beta(1-\gamma)\right]$$

and

$$\Theta \equiv 1 + 2\beta(1-\gamma) - \alpha\left[1 + 2\beta(1-\gamma) + (1-\gamma)(1+\beta)\right]$$

It is easy to prove that $\Gamma < \Theta$. The sign of (8) crucially depends on the size of α relative to those of β and γ, as well as on the value of n. We can define the following critical values of α:

$$\widetilde{\alpha} \equiv \frac{\gamma + \beta(1-\gamma)}{1 + 2\beta(1-\gamma)}$$

$$\hat{\alpha} \equiv \frac{1 + 2\beta(1-\gamma)}{1 + 2\beta(1-\gamma) + (1-\gamma)(1+\beta)}$$

with $\widetilde{\alpha} < \hat{\alpha}$.

Taking the value of n that solves the FOC:

$$\tilde{n} \equiv \frac{\gamma + \beta(1-\gamma) - \alpha\left[1 + 2\beta(1-\gamma)\right]}{z\left[1 + 2\beta(1-\gamma) - \alpha\left[1 + 2\beta(1-\gamma) + (1-\gamma)(1+\beta)\right]\right]}$$

there are three possible cases:

- For a sufficiently small share of capital in income ($\alpha < \widetilde{\alpha}$), both Γ and Θ are positive; hence the MB of children exceeds their MC for $n < \tilde{n}$, while the opposite is the case for $n > \tilde{n}$. This case corresponds to panel (a) of Figure 1.
- For moderate values of α ($\widetilde{\alpha} \leq \alpha \leq \hat{\alpha}$), Θ is positive but Γ is negative; thus the MC always dominates the MB, as can be seen in panel (b).
- If α is high enough ($\alpha > \hat{\alpha}$), both Γ and Θ are negative; however, since $\Gamma < \Theta$ and $zn \leq 1$, the expression will always be negative, as in the previous case. Note that $\hat{\alpha} > 1/2$, so under the typical assumption that the share of capital in income is lower than one half this case is automatically ruled out.

Figure 2 shows the shape of the indirect utility function for the cases differentiated above. As can be observed, there is the possibility of having an interior global maximum under the condition that the share of capital in income is low enough (alternatively, the subjective discount factor is high enough, or the taste for children is high enough). Note that, if there were no taste for children ($\gamma = 0$), these results would not be altered; there would still be an interior global maximum of utility for low values of α and high values of β. So it seems that it is the cost of children that is crucial in determining whether there is an interior solution in the planner's problem.

In Samuelson's exogenous fertility case, panel (b) of the previous figures is the same (with $\gamma = 0$), while for the condition in panel (a) the indirect utility function is always increasing, i.e. the MB dominates the MC for all n[4]. In other words, the condition for having an interior solution with endogenous fertility

[4] In Deardorff's analysis, the indirect utility function in panel (a) is U-shaped, as he assumes capital does not fully depreciate in the production process.

Figure 1: Marginal benefit and marginal cost of children

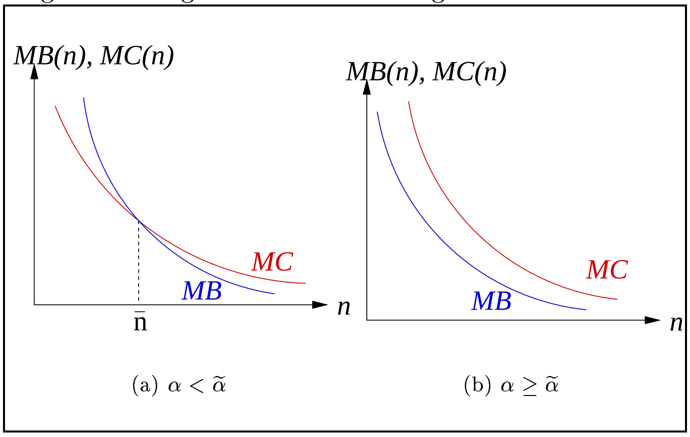
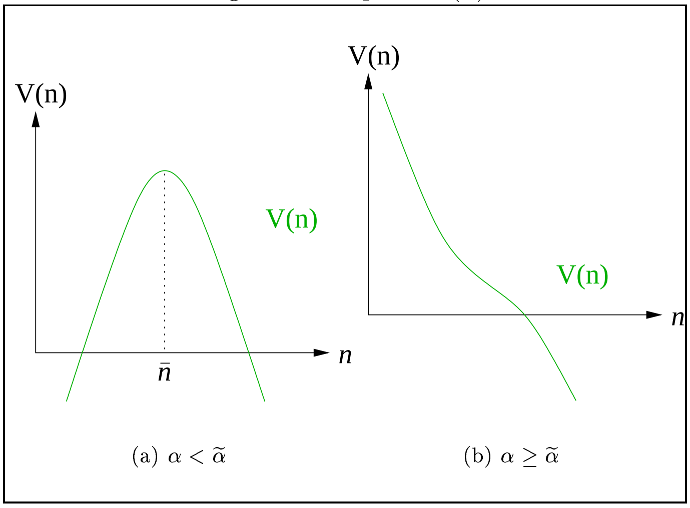


Figure 2: Shape of $V(n)$



is the same as the one for having the ITE dominate the KDE in Samuelson's model: labor must be sufficiently important in the production process and future consumption must not be discounted too much by individuals. Yet, the question remains as to why the utility function eventually decreases under the presence of a cost of children. Recall that both the ITE and the KDE are decreasing in n. When a cost per child is introduced, this cost does not decrease in n as much as the other effects, because we assume that the first child costs the same as the second, and so on[5]. In this way the MC of children eventually dominates for a sufficiently high fertility rate. Thus in the case where the ITE dominates the KDE and hence utility is increasing in n –at least for low values of n– at some point the cost of children will dominate and utility will start decreasing.

[5]The marginal cost of a child in terms of loss in production is decreasing in n due to general equilibrium effects, since this cost is proportional to the per-capita capital stock. However, it is not so decreasing in n as the KDE.

Introducing a constant cost per child is therefore a way of introducing an upper bound for the choice of n, avoiding the solution of having an infinite rate of population growth as the optimal solution for the economy[6].

3 Final Remarks

The previous results would not be altered if the cost of children was introduced as a fixed monetary cost per child. On the other hand, if the cost of children was endogenous –i.e. chosen by the parents– the optimum could again be a corner solution, as parents could choose to have an infinite amount of descendants and to invest a minimum amount of resources on each of them. However, this problem disappears once we consider the human capital that is accumulated when investing in a child's education, with its positive effects on productivity growth[7].

It should not be concluded, from this theoretical analysis, that there exists an optimal population growth rate that each society must try to attain. As Samuelson (1976) claims, "An important purpose of the original analysis was not so much to enable society to identify n^* and normatively to move to it, as to learn what is implied for society's net welfare potentialities by the post-1957 drop in birth rates". After identifying the problem of the non-interiority of the optimal solution, one of his conclusions was that society should "reduce the fears that declining population growth makes old-age security more difficult". Indeed, if n^* were equal to 0, utility would increase as fertility falls, until an infinite level of utility was attained when population tends to disappear. Our analysis with endogenous fertility shows that this result might no longer be true and confirms the conclusion of many other studies developed in the last decades: the demographic transition challenges the future finances of the social security system, and by making the provision of old age consumption more difficult, it may well have negative welfare effects on society.

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[6] The existence of a cost of children imposes an upper limit on the maximum feasible amount of descendants of an individual. Since more children imply less resources available for present consumption, this limit is determined by making consumption in the first period –and hence in the second– equal to zero. Here this maximum number of descendants $n^{\max}$ is given by $1/z$.

[7] See, for example, Peters (1995).

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