Vanderbilt University Department of Economics Working Paper Series
Nashville, TN 37235-1819

Vanderbilt University Department of Economics Working Papers 17-00002

Dominant Strategy Implementability, Zero Length Cycles, and Affine Maximizers

Paul H. Edelman
Vanderbilt University

John A Weymark
Vanderbilt University

Abstract

Necessary conditions for dominant strategy implementability on a restricted type space are identified for a finite set of alternatives. For any one-person mechanism obtained by fixing the other individuals' types, the geometry of the partition of the type space into subsets that are allocated the same alternative is analyzed using difference set polyhedra. Situations are identified in which it is necessary for all cycle lengths in the corresponding allocation graph to be zero, which is shown to be equivalent to the vertices of the difference sets restricted to normalized type vectors coinciding. For an arbitrary type space, it is also shown that any one-person dominant strategy implementable allocation function (i) can be extended to the unrestricted domain and (ii) that it is the solution to an affine maximization problem

Citation: Paul H. Edelman and John A Weymark, (2017) "Dominant Strategy Implementability, Zero Length Cycles, and Affine Maximizers", *Vanderbilt University Department of Economics Working Papers*, VUECON-17-00002.
Contact: Paul H. Edelman - paul.edelman@vanderbilt.edu, John A Weymark - john.weymark@vanderbilt.edu.
Submitted: January 09, 2017. **Published:** January 10, 2017.
URL:http://www.accessecon.com/Pubs/VUECON/VUECON-17-00002.pdf

Dominant Strategy Implementability, Zero Length Cycles, and Affine Maximizers

Paul H. Edelman[a], **John A. Weymark**[b]

[a] *Department of Mathematics and the Law School, Vanderbilt University, 131 21st Avenue South, Nashville, TN 37203-1181, USA.*
E-mail: paul.edelman@law.vanderbilt.edu
[b] *Department of Economics, Vanderbilt University, VU Station B #351819, 2301 Vanderbilt Place, Nashville, TN 37203-1819, USA.*
E-mail: john.weymark@vanderbilt.edu

January 2017

Abstract. Necessary conditions for dominant strategy implementability on a restricted type space are identified for a finite set of alternatives. For any one-person mechanism obtained by fixing the other individuals' types, the geometry of the partition of the type space into subsets that are allocated the same alternative is analyzed using difference set polyhedra. Situations are identified in which it is necessary for all cycle lengths in the corresponding allocation graph to be zero, which is shown to be equivalent to the vertices of the difference sets restricted to normalized type vectors coinciding. For an arbitrary type space, it is also shown that any one-person dominant strategy implementable allocation function (i) can be extended to the unrestricted domain and (ii) that it is the solution to an affine maximization problem.

Journal of Economic Literature classification numbers. D71, D82.

Keywords. Dominant strategy incentive compatibility; implementation theory; mechanism design; Roberts' Theorem; Rockafellar–Rochet Theorem.

1. Introduction

A mechanism consists of an allocation function and a payment function that specify the social alternative that is chosen and the payments (which could be negative) to be made by each individual as a function of their types. We consider an environment in which the number of alternatives is finite and utilities are quasilinear. An individual's type is described by a vector whose components are his valuations of each of the alternatives. Thus, for each alternative, this person's utility is his valuation minus his payment. Types are private information, and so incentives must be provided in order to induce truthful type revelation. A mechanism is dominant strategy incentive compatible if for each type in the domain of the mechanism (the type space), each person can do no better by falsely reporting his type. An allocation rule is dominant strategy implementable if there is a payment function such that the resulting mechanism is dominant strategy incentive compatible. As is well known, such a mechanism can be characterized by the one-person mechanisms that are obtained by considering any individual and fixing the types of the other individuals. Henceforth, we restrict attention to such a one-person mechanism.

In this article, we show that a necessary condition for an allocation function to be dominant strategy implementable is for all cycles in the corresponding allocation graph (defined below) to have zero length when the type space and the allocation function exhibit certain structural features. In particular, all cycle lengths must be zero if the type space is unrestricted. Moreover, all of these cycles have zero length if and only the vertices of particular cones coincide. These cones essentially characterize how the type space is partitioned into types that are assigned the same alternative. For an arbitrary type space, we also show (i) that any dominant strategy implementable one-person allocation function can be extended to the unrestricted type space while preserving dominant strategy implementability and (ii) that this extension can be used to construct a piecewise affine function of the type vector that generates the allocation function by, for each type, maximizing this function over the set of alternatives. The parameters of this piecewise affine function are the average lengths of the arcs that terminate at each node in the allocation graph corresponding to the extended allocation function.

The Rockafellar–Rochet Theorem (Rockafellar, 1970; Rochet, 1987) provides necessary and sufficient conditions for an allocation function to be dominant strategy implementable for an arbitrary type space. For our purposes, the most convenient statement of the Rockafellar–Rochet Theorem is due to Gui et al. (2004) (see also Börgers, 2015; Vohra, 2011). The allocation graph associated with an allocation function is the complete directed graph whose nodes are the set of alternatives and for which the length of the directed arc from alternative a_i to alternative a_j is the infimum of the change in valuation for the individual being considered of having a_j instead of a_i over all all types for which the allocation function chooses a_j.[1] A (directed) cycle in the allocation graph with k arcs is a k-cycle. The Rockafellar–Rochet Theorem says that an allocation function is dominant strategy implementable if and only if all k-cycles in the corresponding allocation graph have nonnegative length for every $k \geq 2$.

Verification of these cycle conditions is impractical when there are many alternatives. Beginning with Bikhchandani et al. (2006), a literature has emerged that has identified a number of restricted multidimensional type spaces for which the nonnegativity of all 2-cycles is sufficient for all cycles in the allocation graph to be nonnegative and, hence, for dominant strategy implementability.[2] Contributions to this literature include Archer and Kleinberg (2014), Ashlagi et al. (2010), Berger et al. (2009), Carbajal and Müller (2015), Kushnir and Galichon (2016), Mishra et al. (2014), and Saks and Yu (2005). For example, the nonnegativity of the 2-cycles has been shown to be sufficient for dominant strategy implementability if the type space is a finite-dimensional convex set (Saks and Yu, 2005) or if it is a multidimensional single-peaked type space (Mishra et al., 2014).

Cuff et al. (2012) show that if the type space is the product of intervals of the real line and a mild regularity condition is satisfied, then (i) a necessary and sufficient condition for dominant strategy implementability is that all 2-cycles in the allocation graph have zero length and (ii) the zero 2-cycle condition implies that all cycles in this graph have zero length.[3] Hence, when their assumptions are satisfied, an allocation function is dominant strategy implementable if and only if all cycles in the allocation graph have zero length. Our results show that the necessity of zero length cycles applies

[1]In our formal analysis, the alternatives are $a_1, \ldots, a_m$ and the node for a_i in the allocation graph is identified with the integer i.

[2]The 2-cycle nonnegativity condition is an analogue for multidimensional type spaces of the monotonicity condition used by Myerson (1981) in his characterization of an optimal auction for a single object.

[3]Their regularity assumption is satisfied if the type space is open (e.g., if the type space is unrestricted).

under much weaker conditions than those identified by Cuff et al. (2012).

In our one-person framework, a dominant strategy implementable allocation function satisfies the revenue equivalence property if any two payment functions that implement the allocation function only differ by a constant. For an arbitrary type space, Heydenreich et al. (2009) show that revenue equivalence is satisfied by a dominant strategy implementable allocation function if and only if the length of the shortest path from any node in the allocation graph to any other node is the negative of the shortest path in the reverse direction. Hence, when there is revenue equivalence, any pair of nodes lies on a zero length cycle, but this cycle may contain more than two arcs. When all 2-cycles have zero length, the shortest path from one node to a second is the direct path between them. Thus, revenue equivalence is satisfied when all 2-cycles have zero length.

An allocation function partitions the type space into sets, each of which consists of the types that are assigned a particular alternative. As shown by Gui et al. (2004) (see also Vohra, 2011), this partition can be identified from a set of polyhedra, one for each alternative, called difference sets. With m alternatives, these difference sets cover $\mathbb{R}^m$ and each pair of these sets has no interior points in common. The facets of the ith difference set are characterized by the lengths in the allocation graph of all the directed arcs that terminate at the node for the ith alternative. The allocation function assigns the ith alternative to any type in the interior of the ith difference set and does not choose this alternative if the type is not in the the ith difference set. Cuff et al. (2012) exploit the geometric structure of the restrictions of the difference sets to the type space in order to establish their results.

We also proceed by investigating the geometric structure of the partition of the type space induced by the allocation function. Restricted to the subspace of $\mathbb{R}^m$ orthogonal to the vector of all 1's, each of the difference sets is a pointed cone. The vertices of these normalized difference sets play a fundamental role in our analysis. Moreover, these vertices can be simply expressed using the arc lengths in the allocation graph. Specifically, we show that the jth component of the vertex of the normalized difference set for the ith alternative is the average length of the directed arcs in the allocation graph that terminate at the ith node minus the length of the arc from node j to node i. For a dominant strategy implementable allocation function, we prove that the vertices of the normalized difference sets coincide if and only if all 2-cycles in the allocation graph have zero length if and only if all cycles in this graph have zero length. We also show that these conditions are necessary for dominant strategy implementability if either (i) the type space has an interior and at least one of these vertices is the projection of a type in the interior of the type space to the subspace orthogonal to the all 1's vector or (ii) there are only two alternatives and the type space is convex.

We show that any one-person allocation function on a restricted type space that is dominant strategy implementable can be extended to a dominant strategy implementable allocation function on all of $\mathbb{R}^m$. This extension is used to show that the original allocation function is the solution to an affine maximization problem. That is, it is an affine maximizer. Specifically, for any type in the domain, for each alternative, we first compute the difference between the valuation for that alternative and the average value of the lengths of all arcs that terminate at its node in the allocation graph corresponding to the extended allocation function. The alternative chosen for this type maximizes the value of this difference. In other words, for each type, the alternative chosen maximizes a piecewise affine function of the type vector over the set of alternatives. For an n-person allocation function, Roberts' Theorem (Roberts, 1979) shows that a necessary condition for a surjective allocation function to be dominant strategy implementable when there are at least three alternatives and the type space is unrestricted is that the chosen alternative maximizes a semipositive weighted sum of the individual valuations and a term that is alternative specific. In contrast, our affine maximization result for a one-person mechanism holds for any type space.

In the first of our two results about the necessity of all cycles being of zero length and in our analysis of universal domain extensions and affine maximizers, it is not assumed that the type space is convex. As a consequence, this article contributes to the relatively small literature on dominant strategy implementability with nonconvex type spaces (see, e.g., Carbajal and Müller, 2015; Kushnir and Galichon, 2016; Mishra et al., 2014).

The rest of this article is organized as follows. In Section 2, we present the model and state the Rockafellar–Rochet Theorem. In Section 3, we introduce difference sets and consider how they are related to the partition of the type space into regions that are allocated the same alternative. In Section 4, we establish some useful results about cycle lengths in the allocation graph. In Section 5, we show that all 2-cycles having zero length is equivalent to the vertices of the normalized difference sets coinciding. In Section 6, we identify sufficient conditions for the necessity of all 2-cycles being of zero length. In Section 7, we consider the prevalence of zero length cycles when not all cycles have zero length. In Section 8, we show how to extend the allocation function to an unrestricted type space. In Section 9, we demonstrate that the allocation function is an affine maximizer. In Section 10, we illustrate our analysis with two economic applications. Finally, in Section 11, we offer some concluding remarks.

2. The Rockafellar–Rochet Theorem

The set of alternatives is $A = \{a_1, \ldots, a_m\}$ for some fixed $m \geq 2$. Let $M = \{1, \ldots, m\}$. An alternative a_i is uniquely identified by the integer i that indexes it. It is sometimes convenient to refer to an alternative by its index rather than as being a member of A.

An individual's type is given by his valuations of each of the alternatives. Because we are considering dominant strategy incentive compatibility, without loss of generality, we can restrict attention to a one-person mechanism in which the types of all but one individual are fixed. Let $v = (v_1, \ldots, v_m) = (v(a_1), \ldots, v(a_m))$ be the *type* of this individual, where $v_i = v(a_i)$ denotes his valuation of the ith alternative. The set of possible types, the *type space*, is V, where $|V| \geq 2$. If $V = \mathbb{R}^m$, the type space is *unrestricted*.

An individual's type is private information, so the mechanism designer must choose an alternative and the payment to be charged (which could be negative) based on the individual's reported type. The reported type could differ from his true type. A *mechanism* is a pair (g, π), where $g\colon V \to A$ is an *allocation function* and $\pi\colon V \to \mathbb{R}$ is a *payment function* that respectively assign an alternative and a payment to each reported type. Utility is quasi-linear in the payment. The *utility* when the true type is v and the reported type is $\tilde{v}$ is $v(g(\tilde{v})) - \pi(\tilde{v})$.[4]

The mechanism designer chooses a mechanism for which the individual always has an incentive to report his true type. An allocation function g is *dominant strategy implementable* if there exists a payment function π such that

$$v(g(v)) - \pi(v) \geq v(g(\tilde{v})) - \pi(\tilde{v}), \quad \forall v, \tilde{v} \in V. \tag{1}$$

If π is such a payment function, then π is said to *implement* g. A mechanism (g, π) for which g is implementable with the payment function π is *dominant strategy incentive compatible.*

[4] In this notation, the true type v is used to value the alternative $g(\tilde{v})$ obtained with the reported type $\tilde{v}$.

Dominant strategy implementability implies that if some alternatives are never chosen, then for any $v, \tilde{v} \in V$ that only differ in the valuations of these alternatives, $v(g(v)) - \pi(v) = v(g(\tilde{v})) - \pi(\tilde{v})$. Because there are no utility consequences if different alternative and payments are chosen for v and $\tilde{v}$, henceforth, we suppose that $g(v) = g(\tilde{v})$ and $\pi(v) = \pi(\tilde{v})$ in such circumstances. With this assumption, the allocation and payment functions only depend on the valuations of the alternatives that are ever chosen. We can therefore reinterpret A as being the subset of alternatives that are chosen for some report of their valuations. With this interpretation of A, g is surjective; that is, $g(V) = A$.

Let

$$R_i = \{v \in V | g(v) = a_i\}, \quad \forall i \in M,$$

be the set of types that are allocated a_i by the allocation function g. These sets partition the set of types. We investigate the geometric structure of this partition and its implications when g is dominant strategy implementable.[5] By construction, $R_i \neq \varnothing$ for all $i \in M$.

For an arbitrary type space, Rochet (1987) has identified a necessary and sufficient condition for an allocation function to be dominant strategy implementable. Rochet's characterization is closely related to the characterization by Rockafellar (1970) of convex functions in terms of their subgradients, so this result is known as the Rockafellar–Rochet Theorem. For our purposes, the most convenient statement of the Rockafellar–Rochet Theorem is provided by the interpretation of this theorem in terms of an allocation graph due to Gui et al. (2004) (see also Vohra, 2011).

The *allocation graph* Γ_g is the complete directed graph that has the set M as the nodes and l_{ij} as the *length* of the directed arc from node i to node j, where

$$l_{ij} = \inf_{v \in R_j} \left[v_j - v_i \right]. \tag{2}$$

In this graph, the length (which could be negative) of the directed arc from i to j is the infimum of the change in the valuation of having alternative a_j instead of alternative a_i over the set of all types for which the allocation function chooses a_j. The length of a loop at node i is $l_{ii} = 0$. The length

[5] Vidali (2009) analyzes the geometric structure of the analogous partition of the type space for a multi-unit auction in which an individual is allocated either 0 or 1 unit of each of m indivisible objects, with the value of his allocation equal to the sum of the valuations of the objects he receives.

l_{ij} is finite for all $i, j \in M$ when g is dominant strategy implementable (see Heydenreich et al., 2009, p. 310).

For any pair of nodes i and j in Γ_g, a *path* from i to j is a sequence of directed arcs $(i_1, i_2), \ldots, (i_{k-1}, i_k)$ for which $i_1 = i$ and $i_k = j$. For any positive integer k, a *k-cycle* is a path with k arcs from i to i. The length of a path or k-cycle is the sum of the lengths of the arcs that comprise it. Changing the starting node without altering the order of the nodes in a k-cycle does not affect its length. As a consequence, if $k = 2$, the two 2-cycles have the same length, so it is not necessary to specify the order in which the nodes appear in a 2-cycle. The allocation function g satisfies the *k-cycle nonnegativity condition* if all k-cycles in Γ_g have nonnegative length and it satisfies the *zero k-cycle condition* if all k-cycles in Γ_g have zero length.

The *Rockafellar–Rochet Theorem* (Rockafellar, 1970; Rochet, 1987) shows that for any type space V, a necessary and sufficient condition for an allocation function g to be dominant strategy implementable is that for any integer $k \geq 2$, the k-cycle nonnegativity condition is satisfied.

Theorem 1. *The following conditions for the allocation function $g \colon V \to A$ are equivalent:*

(i) *g is dominant strategy implementable.*

(ii) *For every integer $k \geq 2$, the k-cycle nonnegativity condition is satisfied.*

3. Difference Sets

As we have seen, the sets $\{R_i\}$ partition of the set of types V, with R_i consisting of all of the types that are assigned the alternative a_i by the allocation function g. The structure of this partition can be identified using what Vohra (2011) calls difference sets. These are polyhedra defined on the universal type space $\mathbb{R}^m$. Before defining these sets, we first introduce some further notation. For each $i, j \in M$, let e_i be the ith coordinate vector and $e_{ij} = e_i - e_j$. For any set $S \subseteq \mathbb{R}^m$, let int$S$ be its interior and relint$S$ be its relative interior.

For all distinct $i, j \in M$, the *pairwise difference set for* (a_i, a_j) is

$$\overline{H}_{ij} = \{v \in \mathbb{R}^m | e_{ij} \cdot v \geq l_{ji}\} \tag{3}$$

and its boundary is

$$H_{ij} = \{v \in \mathbb{R}^m | e_{ij} \cdot v = l_{ji}\}. \tag{4}$$

The pairwise difference set $\overline{H}_{ij}$ is a closed half-space. A type v is in $\overline{H}_{ij}$ if the change in the valuation from obtaining a_i instead of a_j is at least as large as the length of the arc from node j to node i in Γ_g. Recall that this length is the infimum of the change in the valuation from obtaining a_j instead of a_i over all types in V for which g assigns a_j. We let $H_{ii} = \overline{H}_{ii} = \mathbb{R}^m$.

For all $i \in M$, the *difference set for a_i* is

$$P_i = \bigcap_{j=1}^{m} \overline{H}_{ij}. \tag{5}$$

By (3) and (4), P_i is characterized by the lengths of all directed arcs that terminate at node i in the graph Γ_g. The significance of these difference sets is provided by the following theorem, proofs of which may be found in Cuff et al. (2012, Theorem 5) and Vohra (2011, p. 45).

Theorem 2. *For the allocation function $g\colon V \to A$, for any alternative $a_i \in A$:*

(i) *For any type $v \in R_i$, $v \in P_i \cap V$.*

(ii) *If g satisfies the 2-cycle nonnegativity condition, then for any type $v \in \mathrm{int}P_i \cap V$, $v \in R_i$.*

If g is dominant strategy implementable, by Theorem 1, the 2-cycle nonnegativity condition is satisfied. When this is the case, Theorem 2 shows that except for possibly on the boundary of P_i, the set of types in $P_i \cap V$ is R_i. More precisely, any $v \in V$ that is in $\mathrm{int}P_i$ must be assigned the alternative a_i. Conversely, if g assigns a_i to v, then v must be in $P_i \cap V$. Thus, except on their boundaries, the difference sets P_i, $i \in M$, completely identify how alternatives are assigned to types.

Let $\mathbf{1}$ be the all 1's vector and $\mathbf{1}^\perp$ be the subspace of $\mathbb{R}^m$ that is orthogonal to it. In other words, $\mathbf{1}^\perp$ is the set of types whose valuations sum to zero. If $v \in P_i$, then so is $v + \lambda \cdot \mathbf{1}$ for any scalar λ. Thus, each difference set is characterized by its restriction to $\mathbf{1}^\perp$. For all $i \in M$, the *normalized difference set for a_i* is

$$\hat{P}_i = P_i \cap \mathbf{1}^\perp.$$

Note that $\hat{P}_i$ is the orthogonal projection of P_i onto $\mathbf{1}^\perp$. Let

$$\bar{l}_i = \frac{1}{m}\sum_j l_{ji}, \quad \forall i \in M, \tag{6}$$

denote the average length of the arcs in Γ_g that terminate at node i.

For all $m \geq 2$, the normalized difference set $\hat{P}_i$ is a pointed cone. Moreover, its vertex p^i is easily computed from the lengths of the directed arcs in Γ_g that terminate at node i.

Theorem 3. *For all $i \in M$, $\hat{P}_i$ is a pointed cone with vertex p^i whose jth component is*

$$p^i_j = \bar{l}_i - l_{ji}, \quad \forall j \in M. \tag{7}$$

Proof. Because $p^i \in H_{ij}$ for all j,

$$p^i_i - p^i_j = l_{ji}, \quad \forall j \in M. \tag{8}$$

Summing the equations, we obtain

$$np^i_i - \sum_{j\in M} p^i_j = \sum_{j \in M} l_{ji}. \tag{9}$$

Because $p^i \in \mathbf{1}^\perp$,

$$\sum_{j\in M} p^i_j = 0. \tag{10}$$

Hence, (6), (9), and (10) imply that $p^i_i = \bar{l}_i$. Substituting this expression into (8), we obtain (7). □

Thus, p^i_j is the average length of the arcs in Γ_g that terminate at node i minus the length of the arc that goes from node j to node i. Because $l_{ii} = 0$, p^i_i is simply $\bar{l}_i$. As we shall see, the m vertices of the normalized difference sets $\{\hat{P}_i\}$ can be used to characterize the allocation function g.

In determining the implications of dominant strategy implementability, we also make use of the orthogonal projection of the type space V onto $\mathbf{1}^\perp$. This set is denoted by $\hat{V}$, what we call the *projected type space*. In general, $\hat{V}$ is not equal to $V \cap \mathbf{1}^\perp$. For example, if $V = \mathbb{R}^m_+$, then $\hat{V} = \mathbf{1}^\perp$, whereas

$V \cap \mathbf{1}^{\perp} = \mathbf{0}$, where $\mathbf{0} = (0, \ldots, 0)$. If, however, $V = [-d, d]^m$ for some $d > 0$, then $\hat{V} = V \cap \mathbf{1}^{\perp}$.

Except for possibly on the boundaries of the difference sets, a dominant strategy implementable allocation function g can be characterized by an allocation function $\hat{g}$ on $\hat{V}$. To define this function, for each $v \in \hat{V}$, choose any scalar λ_v such that $v + \lambda_v \cdot \mathbf{1} \in V$ and let $\hat{g} \colon \hat{V} \to A$ be defined by

$$\hat{g}(v) = g(v + \lambda_v \cdot \mathbf{1}), \quad \forall v \in \hat{V}. \tag{11}$$

We refer to a vector of the form $v + \lambda \cdot \mathbf{1}$ for some scalar λ as a *translation* of v. It follows from the inequalities in (1) that $\hat{g}$ is dominant strategy implementable. If v and $\tilde{v}$ have the same orthogonal projection on $\hat{V}$, then by Theorem 2, they are assigned the same alternative by g if they are not on the boundary of any the difference sets $\{P_i\}$. When this is the case, the value of $\hat{g}(v)$ does not depend on the choice of λ_v. Thus, $\hat{g}$ can be used to characterize g except for possibly on the boundaries of the difference sets. If it is in fact the case that $g(v) = g(\tilde{v})$ whenever v and $\tilde{v}$ have the same orthogonal projection on $\hat{V}$, then g can be completely recovered from $\hat{g}$ on all of V.

4. Some Cycle Length Lemmas

Cuff et al. (2012) assume that (i) V is the product of intervals of $\mathbb{R}$ and (ii) that the interior of each difference set has a nonempty intersection with the type space.[6] With this structure on V, they show that the zero 2-cycle condition is necessary and sufficient for g to be dominant strategy implementable and that the zero 2-cycle condition implies that all cycles in Γ_g have zero length. In Section 6, we show that all cycles in Γ_g being of zero length is necessary for the dominant strategy implementability of g using much weaker restrictions on the type space than those identified by Cuff et al. (2012). In this section, we present some results about the lengths of cycles in the graph Γ_g that highlight the importance of the 2- and 3-cycles for understanding the implications of dominant strategy implementability.

Consider any nonempty $\mathcal{I} \subseteq M$. The allocation function g satisfies the *2-cycle nonnegativity condition on $\mathcal{I}$* if

$$l_{ij} + l_{ji} \geq 0, \quad \forall i, j \in \mathcal{I}, \tag{12}$$

[6]These assumptions are satisfied if V is the product of open intervals and, hence, are satisfied if the type space is unrestricted.

and it satisfies the *3-cycle nonnegativity condition on $\mathcal{I}$* if

$$l_{ij} + l_{jk} + l_{ki} \geq 0, \quad \forall i, j, k \in \mathcal{I}. \tag{13}$$

If the inequality in (12) (resp. (13)) holds with equality, we have what we call the *zero 2-cycle condition* (resp. *zero 3-cycle condition*) *on $\mathcal{I}$*. If $\mathcal{I} = M$, (12) and (13) are the k-cycle nonnegativity conditions for $k = 2, 3$, respectively.

For any triple of nodes $\mathcal{I}$ in Γ_g, there are six 3-cycles. When g is dominant strategy implementable, these six 3-cycles have zero length if and only the 2-cycles for each pair of nodes in $\mathcal{I}$ also have zero length.

Lemma 1. *For any $\mathcal{I} = \{i, j, k\} \subseteq M$, if the allocation function $g \colon V \to A$ is dominant strategy implementable, then g satisfies the zero 2-cycle condition on $\mathcal{I}$ if and only if it satisfies the zero 3-cycle condition on $\mathcal{I}$.*

Proof. By Theorem 1, g satisfies the 2-cycle and 3-cycle nonnegativity conditions on $\mathcal{I}$. Because addition is associative and commutative,

$$(l_{ij} + l_{ji}) + (l_{jk} + l_{kj}) + (l_{ki} + l_{ik}) = (l_{ij} + l_{jk} + l_{ki}) + (l_{ji} + l_{ki} + l_{kj}). \tag{14}$$

If g satisfies the zero 2-cycle condition on $\{i, j, k\}$, then each of the three bracketed terms on the LHS of (14) and, hence, their sum is 0. By the 3-cycle nonnegativity condition on $\{i, j, k\}$, each of the two bracketed terms on the RHS of (14) is nonnegative. Because their sum is 0, each of these terms must in fact be 0. That is, g satisfies the zero 3-cycle condition on $\{i, j, k\}$. An analogous argument can be used to establish the reverse implication. □

The following result has been established by Cuff et al. (2012, Lemma 2).

Lemma 2. *If the allocation function $g \colon V \to A$ satisfies the zero 2-cycle condition and the 3-cycle nonnegativity condition, then it satisfies the zero k-cycle condition for every integer $k \geq 2$.*

If g is dominant strategy implementable, we know that all cycles in Γ_g have nonnegative length. Hence, an implication of Lemma 2 is that it is sufficient to show that the zero 2-cycle condition holds in order to conclude that all cycles in Γ_g have zero length when g is dominant strategy implementable.

The zero 3-cycle condition fails if there exists any 3-cycle that does not have zero length. This does not imply that a non-zero length cycle can be found that includes any particular node in Γ_g. However, Lemma 3 shows that every node must be part of some non-zero length cycle if the 2- and 3-cycle nonnegativity conditions hold when there exists a 3-cycle that does not have zero length. In particular, this is the case if g is dominant strategy implementable.

Lemma 3. *If the allocation function $g\colon V \to A$ satisfies the 2- and 3-cycle nonnegativity conditions but does not satisfy the zero 3-cycle condition, then for any $i \in M$ there must exist a 3-cycle in Γ_g that includes node i that has positive length.*

Proof. On the contrary, suppose that there exists an i such that for any $j, k \neq i$ we have

$$l_{ij} + l_{jk} + l_{ki} = l_{ji} + l_{ik} + l_{kj} = 0.^{7}$$

Thus,

$$(l_{ij} + l_{ji}) + (l_{ki} + l_{ik}) + (l_{jk} + l_{kj}) = 0.$$

By the 2-cycle nonnegativity condition, each of the bracketed terms in this sum must be nonnegative, and so in fact must be 0. Hence, the zero 2-cycle condition is satisfied on $\{i, j, k\}$. But then by Lemma 1, the zero 3-cycle condition is also satisfied on $\{i, j, k\}$. Because this is true for any $j, k \neq i$, we thus have that the zero 3-cycle condition is satisfied, contradicting the hypothesis that they are not. □

5. Cycle Lengths, Vertices, and Difference Sets

We now investigate the implications for cycle lengths of the relative positions of the boundaries of the pairwise difference sets and of the vertices of the normalized difference sets. We begin with a result showing that the sign of the length of a k-cycle depends on the relative positions of the pairwise difference sets for each pair of nodes in this cycle.

Theorem 4. *For any $\mathcal{I} = \{i_1, \ldots i_{k-1}, i_k\} \subseteq M$, the length of the k-cycle $(i_1, i_2), \ldots, (i_{k-1}, i_k), (i_k, i_1)$ is:*

(i) *zero if there exists a $v \in \cap_{i_j \in \mathcal{I}} H_{i_j i_{j+1}}$.*[8]

[7] Because the length of a 3-cycle only depends on the order of the nodes and not on its initial node, if these equalities hold, then all 3-cycles for the nodes i, j, and k have zero length.

[8] In any k-cycle, addition is modulo k.

(ii) *positive if there exists a* v *that is not in* $\mathrm{int}\overline{H}_{i_j i_{j+1}}$ *for any* $i_j \in \mathcal{I}$ *and that is not in* $\overline{H}_{i_j i_{j+1}}$ *for some* $i_j \in \mathcal{I}$*.*

Proof. The two parts of this theorem follow directly from (4) and (3), respectively. □

The first part of Theorem 4 shows that a k-cycle has zero length if there is a type that is in the boundary of each of the pairwise difference sets for each pair of nodes in the k-cycle. The second part shows that a k-cycle has positive length if the first case does not apply and there is a type that is not in the interior of any of these pairwise difference sets. When $k = 2$, if $H_{ij} = H_{ji}$ coincide (i.e., P_i and P_j share a facet in common), the first case applies and, hence, the 2-cycles for nodes i and j have zero length. Note that if $H_{ij} = H_{ji}$, then $\overline{H}_{ij} \cup \overline{H}_{ji} = \mathbb{R}^m$. If, however, $\overline{H}_{ij} \cup \overline{H}_{ji} \neq \mathbb{R}^m$ (and, therefore, P_i and P_j do not share a common facet), the second case applies and, hence, the 2-cycles for nodes i and j have positive length.

For a dominant strategy implementable allocation function, we now show that the zero 2-cycle condition is equivalent to (i) all of the vertices of the normalized difference sets being equivalent and (ii) the existence of a vector in $\mathbf{1}^\perp$ that is on the boundary H_{ij} of the pairwise difference set $\overline{H}_{ij}$ for each ordered pair of alternatives (a_i, a_j).

Theorem 5. *If the allocation function* $g\colon V \to A$ *is dominant strategy implementable, then the following conditions are equivalent:*

(i) *The vertices* $\{p^i\}$ *of the normalized difference sets coincide.*

(ii) *The simultaneous equations*

$$e_{ij} \cdot v = l_{ji}, \quad \forall i, j \in M, \tag{15}$$

and

$$\mathbf{1} \cdot v = 0 \tag{16}$$

have a solution.

(iii) g *satisfies the zero 2-cycle condition.*

Proof. (i) $\Rightarrow$ (ii). Let p be this common vertex. By the definitions of the normalized difference sets $\{\hat{P}_i\}$, p solves (15) and (16).

(ii) $\Rightarrow$ (i). By definition, a solution to (15) and (16) is a common vertex of all m of the normalized difference sets $\{\hat{P}_i\}$.

(ii) $\Rightarrow$ (iii). Consider any $i \neq j$. The two equations in (15) for i and j imply that $l_{ij} = -l_{ji}$. Thus, the zero 2-cycle condition holds.

(iii) $\Rightarrow$ (ii). The equations in (15) for which $i = j$ are vacuous and so are omitted for the rest of this proof. Without loss of generality, suppose that $i \neq 1 \neq j$. It follows from (8) and (10) in the proof of Theorem 3 that p^1 is the solution to the equations

$$e_{1j} \cdot v = l_{j1} \text{ and } \mathbf{1} \cdot v = 0. \tag{17}$$

Because $e_{ij} = e_{i1} + e_{1j}$ and $e_{i1} = -e_{1i}$, we have

$$\begin{aligned} e_{ij} \cdot p^1 &= e_{i\,1} \cdot p^1 + e_{1\,j} \cdot p^1 \\ &= -e_{1\,i} \cdot p^1 + e_{1\,j} \cdot p^1 \\ &= -l_{i1} + l_{j1}, \end{aligned}$$

where the last equality follows from (17). By assumption, g satisfies the zero 2-cycle condition, and so $l_{j1} = -l_{1j}$. By Lemma 2, g also satisfies the zero 3-cycle condition and, therefore, $l_{ji} = -l_{i1} - l_{1j}$. Thus,

$$e_{ij} \cdot p^1 = -l_{i1} + l_{j1} = -l_{i1} - l_{1j} = l_{ji},$$

which shows that p^1 is a solution to (15) and (16). □

Example 1. For the three alternative case, the situation in which the conditions in Theorem 5 are satisfied is illustrated in Figure 1.[9] The common vertex of the three normalized difference sets $\hat{P}_1$, $\hat{P}_2$, and $\hat{P}_3$ is p. Because the allocation function g is surjective, each of these three sets must have a nonempty intersection with the projected type space $\hat{V}$ and each type in $\hat{V}$ must be in at least one of these three projected difference sets. The vertex p need not be in $\hat{V}$. If $\hat{V}$ is not convex, it is possible that $\hat{P}_i \cap \hat{V}$ is not connected for some values of i. In the diagram, this is the case for $i = 1$. Because p is a common vertex of $\hat{P}_1$, $\hat{P}_2$, and $\hat{P}_3$, it must be on the boundaries of every pairwise difference set, which by Theorem 4 implies that all 3-cycles have

[9]In our diagrams, the orientation is chosen so that $\mathbf{1}^\perp$ lies flat in the page.

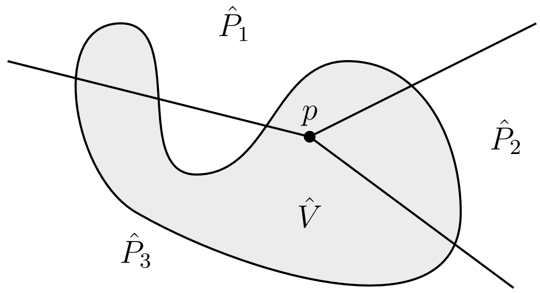

Figure 1: Satisfaction of the Conditions in Theorem 5.

zero length. It then follows from Lemma 1 that all 2-cycles have zero length as well, which by Theorem 4 is only possible if each projected difference set has a facet in common with each of the other two of these sets.

Example 2. Figure 2 provides an illustration of the failure of the conditions in Theorem 5 when there are three alternatives. In the diagram, the vertex p^2 of $\hat{P}_2$ has been chosen to lie outside of $\hat{V}$ and to differ from the vertices p^1 of $\hat{P}_1$ and p^3 of $\hat{P}_3$. For each $i, j \in M$, it follows from the definition of H_{ij} in (4) that $H_{ij} \cap \mathbf{1}^\perp$ is a line when $m = 3$. On this line, $v_i - v_j = l_{ji}$. Thus, the line $H_{ij} \cap \mathbf{1}^\perp$ is parallel to the line $H_{ji} \cap \mathbf{1}^\perp$. In the diagram, these observations are reflected by the fact that any normalized difference set $\hat{P}_i$ has a facet with the same slope as one of the facets of $\hat{P}_j$ for $j \neq i$. Because $\hat{P}_1$ and $\hat{P}_3$ have no type in common, the length of a 2-cycle for nodes 1 and 3 is positive. Because the allocation function g is surjective, each of the other two pairs of normalized difference sets must share a common facet, and so the lengths of the 2-cycles for the node pairs $\{1, 2\}$ and $\{2, 3\}$ are zero.

Let $\hat{F}_{13}$ and $\hat{F}_{31}$ denote the parallel facets of $\hat{P}_1$ and $\hat{P}_3$, respectively. $\hat{F}_{13}$ lies on $H_{13} \cap \mathbf{1}^\perp$ and $\hat{F}_{31}$ lies on $H_{31} \cap \mathbf{1}^\perp$. The value of $v_1 - v_3$ (and, hence, of $v_3 - v_1$) is constant on lines that are parallel to these two lines. For any type v that is a translation of a type in $\text{int}\hat{P}_1 \cap \hat{V}$, we know that $g(v) = a_1$ and for any type v that is a translation of a type that is not in $\hat{P}_1 \cap \hat{V}$, we know that $g(v) \neq a_1$. Hence, because the value of l_{31} is chosen to be the infimum of $v_1 - v_3$ for all types v for which $g(v) = a_1$, it follows that the value of $v_1 - v_3$ is decreasing as we move to the right in Figure 2 and that the vertex p^1 is the rightmost boundary point of $\hat{P}_1 \cap \hat{V}$. Analogous reasoning shows that p^3

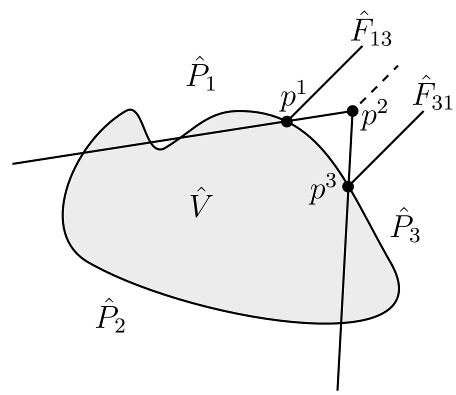

Figure 2: Failure of the Conditions in Theorem 5.

is the uppermost boundary point of $\hat{P}_3 \cap \hat{V}$.[10]

In our discussion of Example 2, we have seen that it is possible that some of the 2-cycles do not have zero length. In this three alternative example, only 2-cycles involving one particular node have a non-zero length. These observations are a consequence of a more general feature of dominant strategy implementable allocation functions that is discussed in Section 7.

6. The Necessity of Zero Length Cycles

In Example 1, all cycles in the allocation graph have zero length. In this section, we identify circumstances in which dominant strategy implementability requires this to be the case.

Suppose that the allocation function g satisfies the 2-cycle nonnegativity condition, which is the case if g is dominant strategy implementable. Also suppose that the type space V has an interior. Consider any type v in the interior of V. The following lemma shows that that if v is in the relative interior of the facet of the difference set for alternative a_i that is identified by the boundary of the pairwise difference set for (a_i, a_j), then the lengths of the 2-cycles in Γ_g for nodes i and j are zero.

Lemma 4. *For all $i, j \in M$, if the allocation function $g\colon V \to A$ satisfies the 2-cycle nonnegativity condition, the type space V has a nonempty interior,*

[10]The significance of the dashed line in Figure 2 is discussed in Sections 8 and 9.

and there exists a $v \in P_i \cap \text{int}V$ *for which* $v \in \text{relint}(H_{ij} \cap P_i)$*, then* $l_{ij} + l_{ji} = 0$.

Proof. By assumption, $l_{ij} + l_{ji} \geq 0$. Contrary to the lemma, suppose that $l_{ij} + l_{ji} = \delta > 0$. Let B be an open ball containing v for which (i) $B \subset \text{int}V$ and (ii) $B^* = B \cap \text{relint}(H_{ij} \cap P_i)$ is open and of co-dimension 1. We first show that there must be a type $v^* \in B^*$ that is not contained in P_k for any $k \neq i$. We then show that there is a type in V that is not in any of the P_k, which contradicts the surjectivity of g.

The requirement that $B^* \subset \text{int}(H_{ij} \cap P_i)$ implies that for all $v^* \in B^*$,

$$e_{ij} \cdot v^* = l_{ji} \tag{18}$$

and

$$e_{ik} \cdot v^* > l_{ki}, \quad \forall k \neq i.$$

It then follows that $B^* \cap P_j = \varnothing$ because for any $v^* \in B^*$,

$$e_{ji} \cdot v^* = -e_{ij} \cdot v^* = -l_{ji} = l_{ij} - \delta < l_{ij}.$$

For every $k \neq i, j$, we know that $\text{int}P_k \cap \text{int}P_i = \varnothing$. Hence, if $P_k \cap B^* \neq \varnothing$, then $P_k \cap B^*$ lies in the boundary of P_k. Consequently, $P_k \cap B^*$ cannot lie in a facet of P_k because that would imply that P_k has a facet parallel to H_{ij}, which only happens when $k = j$. It then follows that for each $k \neq i$, $P_k \cap B^*$ can have at most co-dimension 2. Thus, $B^* - \cup_{k \neq i}(P_k \cap B^*) \neq \varnothing$.

Consider any $v^* \in B^* - \cup_{k \neq i}(P_k \cap B^*)$. The function $d(\lambda) = v^* + \lambda e_{ji}$ for $\lambda \geq 0$ defines a ray with origin v^* in the direction e_{ji}. For any $\lambda > 0$,

$$e_{ij} \cdot d(\lambda) = e_{ij} \cdot v^* + \lambda e_{ij} \cdot e_{ji} = l_{ji} - 2\lambda < l_{ji},$$

where the second equality follows from (18). But by (3), in order to have $d(\lambda) \in P_i$, it must be the case that $e_{ij} \cdot d(\lambda) \geq l_{ji}$. Hence, $d(\lambda) \notin P_i$. Because each of the P_k is closed, $v^* \in \text{int}V$, and $v^* \notin P_k$ for any $k \neq i$, there must be a value $\lambda' > 0$ sufficiently close to 0 for which $d(\lambda') \in V$ such that $d(\lambda') \notin P_k$ for any $k \neq i$. Thus, $d(\lambda') \in V$, but $d(\lambda') \notin P_k$ for any k, which contradicts the surjectivity of g. Hence, $l_{ij} + l_{ji} = 0$. □

We now use Lemma 4 to prove the following theorem.

Theorem 6. *If the allocation function* $g \colon V \to A$ *is dominant strategy implementable, the type space* V *has a nonempty interior, and there exists an* $i \in M$ *such that the vertex* p^i *of the normalized difference set* $\hat{P}_i$ *is the projection of some type* $v^i \in \text{int}V$ *onto* $\mathbf{1}^\perp$*, then*

(i) *the vertices* $\{p^j\}$ *of the normalized difference sets are identical.*

(ii) g *satisfies the zero* k*-cycle condition for all* $k \geq 2$.

Proof. We first prove (i). Because g is dominant strategy implementable, by Theorem 5, we know that the vertices $\{p^j\}$ of the normalized difference sets are identical if and only g satisfies the zero 2-cycle condition. Moreover, by Lemma 1, the zero 2-cycle condition holds if and only if the zero 3-cycle condition does as well.

Suppose that the $\{p^j\}$ are not all identical. Then some zero 3-cycle condition fails. Consider any $i \in M$. By Lemma 3, there exist $j, k \neq i$ such that some 3-cycle using i, j, and k as nodes has positive length. Without loss of generality, suppose that

$$l_{ki} + l_{ij} + l_{jk} > 0.$$

Because p^i is the projection onto $\mathbf{1}^\perp$ of some type $v^i \in \text{int}V$, there exist $v^j, v^k \in P_i \cap \text{int}V$ for which $v^j \in \text{relint}(H_{ij} \cap P_i)$ and $v^k \in \text{relint}(H_{ik} \cap P_i)$. By Theorem 1, g satisfies the 2-cycle nonnegativity condition. Therefore, the assumptions of Lemma 4 are satisfied, and so $l_{ij} + l_{ji} = 0$ and $l_{ik} + l_{ki} = 0$. Hence,

$$l_{jk} > l_{ji} + l_{ik}.$$

Choose $\epsilon > 0$ so that $2\epsilon < l_{jk} - (l_{ji} + l_{ik})$ and consider the type

$$\hat{v} = p^i + \epsilon e_k - \epsilon e_j.$$

We know from Theorem 3 that $p^i_j = \bar{l}_i - l_{ji}$ for all $j \in M$. Because $l_{ii} = 0$, a simple computation shows that

$$e_{ik} \cdot \hat{v} = l_{ki} - \epsilon < l_{ki}.$$

Hence, $\hat{v} \notin \hat{P}_i$.

We also have that

$$\begin{aligned} e_{jk} \cdot \hat{v} &= (-e_{ij} + e_{ik}) \cdot \hat{v} \\ &= -l_{ji} - \epsilon + l_{ki} - \epsilon \\ &= -l_{ji} - l_{ki} - 2\epsilon \\ &\leq l_{kj} - 2\epsilon \\ &< l_{kj}, \end{aligned}$$

where the third equality follows from the fact that $l_{ik} = -l_{ki}$ and the first inequality follows from the fact that 3-cycles have nonnegative length. Hence, $\hat{v} \notin \hat{P}_j$.

Similarly,

$$\begin{aligned} e_{kj} \cdot \hat{v} &= (e_{ki} + e_{ij}) \cdot \hat{v} \\ &= l_{ik} + \epsilon + l_{ji} + \epsilon \\ &= -l_{ki} - l_{ij} + 2\epsilon \\ &< l_{jk}, \end{aligned}$$

where the third equality follows from the fact that both $l_{ij} = -l_{ji}$ and $l_{ik} = -l_{ki}$ and the inequality follows from the fact that 3-cycles have nonnegative length. Hence, $\hat{v} \notin \hat{P}_k$.

For any $h \neq i, j, k$,

$$\begin{aligned} e_{hk} \cdot \hat{v} &= (e_{hi} + e_{ik}) \cdot \hat{v} \\ &= l_{ih} + l_{ki} - \epsilon \\ &= -l_{hi} - l_{ik} - \epsilon \\ &\leq l_{kh} - \epsilon \\ &< l_{kh}, \end{aligned}$$

where the third equality follows from the fact that both $l_{ih} = l_{hi}$ and $l_{ik} = -l_{ki}$ and the first inequality follows from the fact that 3-cycles have nonnegative length. Hence, $\hat{v} \notin \hat{P}_h$ for any $h \neq i, j, k$.

By Theorem 2, we have thus shown that $\hat{v} \notin V$. However, by choosing ϵ sufficiently small, $\hat{v} \in \hat{V}$, which is a contradiction.

Part (ii) of the theorem now follows immediately from Part (i), Lemma 2, and Theorem 5. □

Figure 1 can be used to provide some intuition for Theorem 6. For concreteness, consider the normalized difference set $\hat{P}_1$ for alternative a_1. It vertex is $p^1 = p$. The upward sloping line emanating from p is the facet of $\hat{P}_1$ that is defined using the pairwise difference set for (a_1, a_2). This facet must be parallel to the corresponding facet of $\hat{P}_2$. If these facets do not coincide, because the interiors of $\hat{P}_1$ and $\hat{P}_2$ have an empty intersection, there must be points near p that lie between these two facets. Any such point must be the projection of a type in V that is not in any difference set and, hence, not

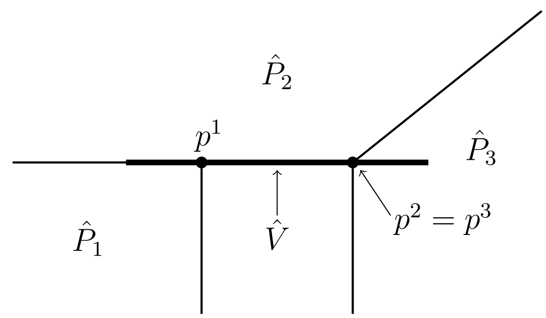

Figure 3: Failure of the Conditions in Theorem 6.

assigned any alternative, which is inconsistent with the surjectivity of the allocation function. Thus, $\hat{P}_1$ and $\hat{P}_2$ share a common boundary. Analogously, $\hat{P}_1$ and $\hat{P}_3$ also share a common boundary. In order for $\hat{P}_2$ and $\hat{P}_3$ to be cones, it then follows that p is a common vertex of these two normalized difference sets, from which it follows that they share a common boundary. When a pair of normalized difference sets share a common boundary, any 2-cycle for the corresponding nodes in Γ_g has zero length. Thus, in Example 1, all 2-cycles and, hence, all cycles have zero length.

In Theorem 6, it is not assumed that the type space is convex. Thus, this theorem contributes to our understanding of the implications of dominant strategy implementability in nonconvex environments, thereby complementing the analyses of nonconvex type spaces by Carbajal and Müller (2015), Kushnir and Galichon (2016), and Mishra et al. (2014).

The assumption in Theorem 6 that the type space has an interior cannot be dispensed with, as the following example demonstrates.

Example 3. Suppose that there are three alternatives and that $V = \hat{V}$, as illustrated in Figure 3. All types weakly to the left of p^1 are assigned a_1, all types weakly to the right of $p^2 = p^3$ are assigned a_3, and the types between these vertices are assigned a_2. While the length of a 2-cycle for nodes 1 and 2 and for nodes 2 and 3 are both zero, the length of a 2-cycle for nodes 1 and 3 is positive.

An essential feature of Example 3 is that there are more than two alternatives. Theorem 7 shows that if there are only two alternatives and the type space is convex, then it is necessary for a dominant strategy implementable allocation function to satisfy the zero 2-cycle condition or, equivalently, for all of the vertices of the normalized difference sets to coincide. In this theorem, the type space is not required to have an interior.

Theorem 7. *If $m = 2$, the allocation function $g\colon V \to A$ is dominant strategy implementable, and the type space V is convex, then*

(i) *the vertices $\{p^j\}$ of the normalized difference sets coincide.*

(ii) *g satisfies the zero 2-cycle condition.*

Proof. Because g is surjective, $V \cap P_i \neq \varnothing$, $i = 1, 2$. Hence, because V is convex, if $l_{12} + l_{21} > 0$, there exist types in V that are in neither difference set, which by Theorem 2 is not possible. Thus, (ii) holds. It then follows that the two normalized difference set (which are rays) share a single point in common, which establishes (i).[11] □

As is the case when the assumptions of Theorem 6 hold, the failure of the zero 2-cycle condition implies that some types are not assigned any alternative, thereby violating the surjectivity of the allocation function.

The assumptions of Theorem 7 are satisfied by a Vickrey (1961) auction of a single indivisible good. For such an auction, there are two possible outcomes: the individual is either awarded the good or he is not. The type space is one-dimensional convex set because the value of not receiving the good is 0 and the value of receiving the good can take on any value in an interval of $\mathbb{R}$.[12]

Theorem 7 does not generalize to more than two alternatives even when the type space has an interior. When there are three alternatives, this can be shown by considering a variant of Example 2.

Example 4. If the type space V is convex, then so is the projected type space $\hat{V}$. By an appropriate choice of V, it is possible to have the same normalized difference sets as in Figure 2. In particular, it is possible for p^1 and p^3 to lie in the boundary of $\hat{V}$ and for p^2 to lie outside $\hat{V}$. In this example, a 2-cycle for nodes 1 and 3 has positive length.

[11] Part (ii) of this theorem has been established by Cuff et al. (2012, p. 384). We include its short proof here because it provides the intuition for why this result is true.

[12] Vickrey auctions are discussed in more detail in Section 10.

7. The Ubiquity of Zero Length 2-Cycles

In this section, we consider the prevalence of zero length 2-cycles when the allocation function is dominant strategy implementable. In Examples 1, 2, and 3, all of the 2-cycles have zero length except for at most one pair of alternatives. As we shall see, this must be the case when the type space is connected when, as in these examples, there are three alternatives. Regardless of the number of alternatives, provided that the type space is connected, we show that there must be many zero length cycles.

The pairs of alternatives for which 2-cycle lengths are zero can be described using a graph. Formally, for the allocation function $g\colon V \to A$, the *zero 2-cycle graph* is the graph Γ_g^2 with node set M that has an edge between nodes i and j, denoted $i \sim j$, if and only if $l_{ij} + l_{ji} = 0$.[13]

Following Vohra (2011, p. 63), the allocation graph Γ_g is said to be *2-cycle connected* if Γ_g^2 is connected. If Γ_g is 2-cycle connected, then for any partition of the node set M of Γ_g into two non-empty sets $\mathcal{I}$ and $\mathcal{J}$, there must be an $i \in \mathcal{I}$ and a $j \in \mathcal{J}$ such that the 2-cycles for these nodes have zero length. Because Γ_g^2 has m nodes, it must have at least $m-1$ edges, and therefore there must be at least $m-1$ pairs $\{i, j\}$ of distinct nodes in M whose 2-cycles have zero length if Γ_g is 2-cycle connected. Thus, if Γ_g is 2-cycle connected, because there are two 2-cycles for each pair of nodes, there must be at least $2m-2$ zero length cycles when Γ_g is 2-cycle connected. Informally, zero length 2-cycles are ubiquitous.

If the assumptions of Theorem 6 are satisfied, then Γ_g^2 is a complete graph and, hence, Γ_g is 2-cycle connected. The allocation graph is also 2-cycle connected if the type space is connected when the allocation function is dominant strategy implementable.[14]

Theorem 8. *If the allocation function $g\colon V \to A$ is dominant strategy implementable and the type space V is connected, then Γ_g is 2-cycle connected.*

Proof. By identifying the set T in Theorem 4.3.4 in Vohra (2011) with the type space V, the assumptions of Vohra's theorem are satisfied. In the proof of his theorem, Vohra shows that Γ_g is 2-cycle connected. □

[13] A formally equivalent graph is used by Kushnir and Galichon (2016) to study the sufficiency of the 2-cycle nonnegativity condition for dominant strategy implementability.

[14] Note that it is not assumed in Theorem 6 that V is connected.

The assumptions of Theorem 8 are satisfied for Examples 1, 2, and 3. In each of these examples, $m = 3$, and so there must be at least two pairs of nodes for which the 2-cycles have zero length, which is indeed the case.

Vohra (2011, Lemma 4.3.3) shows that a dominant strategy implementable allocation function satisfies the revenue equivalence property if the allocation graph is 2-cycle connected. As the following example demonstrates, the converse does not hold. That is, revenue equivalence and dominant strategy implementability do not imply that the allocation graph is 2-cycle connected.

Example 5. Let $m = 3$ and suppose that with the allocation function g, $l_{12} = l_{13} = l_{32} = 2$ and $l_{21} = l_{31} = l_{23} = -1$. All cycles in Γ_g have nonnegative length, so g is dominant strategy implementable. The shortest path from node 1 to node 2 has length 2 and the shortest path in the reverse direction has length -2. The shortest path from node 1 to node 3 has length 1 and the shortest path in the reverse direction has length -1. The shortest path from node 2 to node 3 has length -1 and the shortest path in the reverse direction has length 1. In each case, the sum of the two lengths is 0, so revenue equivalence is satisfied. However, as is readily verified, none of the 2-cycles has zero length.

8. Extending the Domain

The allocation function $g^+ : \mathbb{R}^m \to A$ is a *universal domain extension* of the allocation function $g : V \to A$ if $g^+(v) = g(v)$ for all $v \in V$. In this section, we show that any dominant strategy allocation function on a restricted type space has a universal domain extension that is also dominant strategy implementable.

Before turning to our formal analysis, we offer some intuition for the existence of a universal domain extension by reconsidering Examples 1, 2, and 3. In Figure 1, the union of the three normalized difference sets $\{\hat{P}_i\}$ is all of $\mathbf{1}^\perp$ and, hence, the union of the the corresponding difference sets $\{P_i\}$ is all of $\mathbb{R}^m$. As a consequence, the allocation function for this example has a universal domain extension. For all $i, j \in M$, this extension assigns alternative a_i to any $v \in \text{int}P_i$, a_i or a_j to any $v \in P_i \cap P_j$, and a_1, a_2, or a_3 to any $v \in P_1 \cap P_2 \cap P_3$. In Figures 2 and 3, the union of the three normalized difference sets $\{\hat{P}_i\}$ is a strict subset of $\mathbf{1}^\perp$. Therefore, the union of the corresponding difference sets $\{P_i\}$ does not cover $\mathbb{R}^m$. Nevertheless, the allocation functions for these examples also have a universal domain extension.

In Figure 2, by moving the facets $\hat{F}_{13}$ and $\hat{F}_{31}$ so that they coincide with the dashed line in the diagram, the resulting normalized difference sets cover $\mathbf{1}^{\perp}$ and their intersections with the normalized type space $\hat{V}$ are unchanged. In Figure 3, if the vertex p^1 is moved to the right so that it coincides with p^2 and p^3, the intersections of the interiors of the resulting normalized difference sets with the normalized type space are unchanged. Using the corresponding modified difference sets on all of $\mathbb{R}^m$, the requisite extension can be constructed as in the first example. Note that the arc lengths that define the difference sets that have been modified are not the lengths that define the corresponding difference sets for the original type space. In each of these cases, the constructed allocation function on the unrestricted type space is dominant strategy implementable.

The inequalities (1) that define dominant strategy implementability imply that the payments must be the same for any types that are allocated the same alternative. Thus, a payment function π that implements the allocation function g can be equivalently described in terms of a function ρ_g that assigns a value to each node in the corresponding allocation graph Γ_g. Heydenreich et al. (2009, Observation 1) have shown that π implements g if and only if ρ_g is a node potential.

Formally, for the allocation function $g\colon V \to A$, the function $\rho_g\colon M \to \mathbb{R}$ is a *node potential* if

$$\rho_g(j) \leq \rho_g(i) + l_{ij}, \quad \forall i, j \in M. \tag{19}$$

That is, a node potential assigns a scalar to each node in the graph Γ_g in such a way that (19) holds. The payment function $\pi\colon V \to \mathbb{R}$ *corresponds to the node potential* ρ_g if for all $i \in M$ and all $v \in V$ for which $g(v) = a_i$, $\pi(v) = \rho_g(i)$. In other words, the payment required by the payment function π for any type $v \in V$ that the allocation function g assigns a_i is the value assigned to the ith node in Γ_g by the node potential ρ_g. Theorem 9 provides a formal statement of the Heydenreich et al. (2009) characterization of dominant strategy incentive compatibility in terms of node potentials.

Theorem 9. *For the allocation function $g\colon V \to A$ and payment function $\pi\colon V \to \mathbb{R}$, (g, π) is dominant strategy incentive compatible if and only if π corresponds to a node potential $\rho_g\colon M \to \mathbb{R}$.*

Consider any dominant strategy implementable allocation function g and let π be a payment function that implements it. By Theorem 9, π corresponds to some node potential ρ_g. Let

$$l_{ij}^+ = \rho_g(j) - \rho_g(i), \quad \forall i, j \in M. \tag{20}$$

The value l_{ij}^+ is the increment in the payment required if a_j is chosen instead of a_i by the allocation function g using the payment function π corresponding to the node potential ρ_g. The *node potential allocation graph* Γ_g^+ is defined to be the complete directed graph with node set M for which the length of the directed arc from node i to node j is l_{ij}^+.

It follows immediately from (20) that every cycle in Γ_g^+ has zero length.

Lemma 5. *If $\rho_g\colon M \to \mathbb{R}$ is a node potential for the dominant strategy implementable allocation function $g\colon V \to A$, then for every integer $k \geq 2$, any k-cycle in the node potential allocation graph Γ_g^+ has zero length.*

Lemma 6 shows that the length of any arc in the allocation graph Γ_g is at least as large as the length of the corresponding arc in the node potential allocation graph Γ_g^+ and that these arc lengths coincide when an arc is part of a zero length 2-cycle of Γ_g.

Lemma 6. *If $\rho_g\colon M \to \mathbb{R}$ is a node potential for the dominant strategy implementable allocation function $g\colon V \to A$, then for all $i, j \in M$,*

$$l_{ij} \geq l_{ij}^+. \tag{21}$$

and for all $i, j \in M$ for which $l_{ij} + l_{ji} = 0$,

$$l_{ij}^+ = l_{ij}. \tag{22}$$

Proof. Because ρ_g is a node potential for g, (21) follows from (19) and (20). Consider any $i, j \in M$ for which $l_{ij} + l_{ji} = 0$. Because $l_{ij} + l_{ji} = 0$ and $l_{ij}^+ + l_{ji}^+ = 0$, if $l_{ij} > l_{ij}^+$, we would have

$$0 = l_{ij} + l_{ji} > l_{ij}^+ + l_{ji}^+ = 0,$$

which is impossible. Hence, because (21) holds, (22) does as well. □

For all $i \in M$, let P_i^+ be the pairwise difference set for a_i defined as in (5) but using the lengths $\{l_{ij}^+\}$ instead of the lengths $\{l_{ij}\}$ when defining the analogues of the pairwise difference sets in (3). Also let $\hat{P}_i^+ \subseteq \mathbf{1}^\perp$ be the corresponding normalized difference set for a_i. An implication of Lemma 6 is that for all $i \in M$, $P_i \subseteq P_i^+$ and $\hat{P}_i \subseteq \hat{P}_i^+$. In moving from P_i to P_i^+, any facet of P_i that is defined using an alternative whose node forms a 2-cycle of Γ_g^2 with node i is unchanged, whereas any facet of P_i that is defined using an alternative whose node does not form a 2-cycle of Γ_g^2 with node i is moved parallel so as to increase the size of this difference set. We use these observations to prove that any dominant strategy implementable allocation function has a universal domain extension that is also dominant strategy implementable.

Theorem 10. *If the allocation function $g\colon V \to A$ is dominant strategy implementable, then g has a universal domain extension $g^+\colon \mathbb{R}^m \to A$ that is dominant strategy implementable.*

Proof. Because g is dominant strategy implementable, by Theorem 9, there exists a node potential $\rho_g\colon M \to \mathbb{R}$ and a payment function $\pi\colon V \to A$ corresponding to it that implements g. By Lemma 6, $l_{ij}^+ = l_{ij}$ and $l_{ji}^+ = l_{ji}$ for any pair of nodes i and j for which $i \sim j$ in the 2-cycle graph Γ_g^2. For any pair of nodes i and j for which $i \not\sim j$, by (21), $l_{ij} > l_{ij}^+$ and $l_{ji} > l_{ji}^+$. Hence, by the definitions of P_i and P_i^+,

$$P_i \subseteq P_i^+, \quad \forall i \in M. \tag{23}$$

We now show that

$$\cup_{i \in M} P_i^+ = \mathbb{R}^m. \tag{24}$$

On the contrary, suppose that there exists a $v \in \mathbb{R}^m$ for which $v \notin P_i^+$ for any $i \in M$. Using the lengths $\{l_{ij}^+\}$ instead of the lengths $\{l_{ij}\}$ in (3) and (5), it then follows that for all $i \in M$, there exists an $i_j \in M$ such that

$$v_i - v_{i_j} < l_{i_j i}^+. \tag{25}$$

Because the number of nodes is finite, there exists a k-cycle for some $k \in \{2, \ldots, M\}$ in which each arc is the arc from i to i_j for some i. Let E be the set of the arcs in this cycle with the arc which starts at node i denoted by ii_j. By (25),

$$0 = \sum_{ii_j \in E} [v_i - v_{i_j}] < \sum_{ii_j \in E} l_{ii_j}^+. \tag{26}$$

By Lemma 5, every cycle in the complete graph Γ_g^+ has zero length, which contradicts (26). Hence, (24) holds.

We now construct the allocation function $g^+ \colon \mathbb{R}^m \to A$. For all $v \in V$, we let $g^+(v) = g(v)$ so that g^+ is a universal domain extension of g. By construction, $\mathrm{int}P_i^+ \cap \mathrm{int}P_j^+ = \varnothing$ for all $i, j \in M$. For all $i \in M$, let $g^+(v) = a_i$ for any $v \in \mathrm{int}P_i^+ \setminus V$. For any other $v \in \mathbb{R}^m$, there exists a maximal subset $\mathcal{I} \subseteq M$ for which $v \in \cap_{I \in \mathcal{I}} P_i^+$. For such a v, let $g^+(v) = a_i$ for some $i \in \mathcal{I}$. By construction, the allocation function g^+ satisfies the conditions in Theorem 2 reinterpreted so as to apply to g^+.

By Lemma 5, all cycles in Γ_g^+ have zero length. Hence, by Theorem 1, g^+ is dominant strategy implementable. □

An implication of Theorem 10 is that Γ_g^+ is the allocation graph for the allocation function g^+. Because all 2-cycles in this graph have zero length and g^+ is dominant strategy implementable, it follows from Theorem 5 that the normalized difference sets $\{\hat{P}_i^+\}$ have a common vertex, which we denote by p^+.

In Figure 1, $p^+ = p$, in Figure 2, $p^+ = p^2$, and in Figure 3, $p^+ = p^2 = p^3$. In Example 6, we illustrate Theorem 10 with a three alternative example in which p^+ does not coincide with a vertex of any of the normalized difference sets for g.

Example 6. The projected type space $\hat{V}$ and the three normalized difference sets $\hat{P}_1$, $\hat{P}_2$, and $\hat{P}_3$ are as illustrated in Figure 4. Note that $\hat{V}$ is not connected. Because $l_{12} + l_{21} = 0$, p^+ must lie on the line through p^1 and p^2. It must also lie on a line that is parallel to the upward sloping facet of $\hat{P}_1$ and on a line that is parallel to the upward sloping facet of $\hat{P}_2$. The vertex p^+ must also lie weakly to the right of p^1 and weakly to the left of $\hat{P}_3$. Its exact location on the line through p^1 and p^2 depends on which payment function is used to implement g or, equivalently, what node potential is used.[15] The rays that originate at p^+ are the facets that separate pairs of the normalized difference sets $\{\hat{P}_i^+\}$.[16]

As we have noted, there is not a unique universal domain extension in Example 6. We now show that a dominant strategy implementable allocation

[15]In this example, we have a failure of revenue equivalence. That is, there exist payment functions π and π' that both implement g for which there is no constant c such that $\pi(v) = \pi'(v) + c$ for all $v \in V$.

[16]To avoid clutter, we do not label these sets in the diagram. They can be inferred from the fact that $\hat{P}_i \subseteq \hat{P}_i^+$ for all $i \in M$.

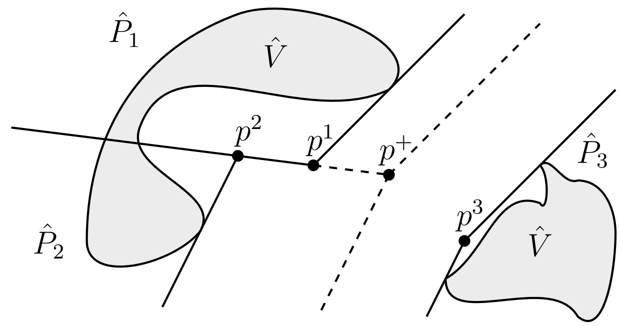

Figure 4: Extending g to g^+.

function g has a unique universal domain extension if the zero 2-cycle graph Γ_g^2 is connected. This graph need not have any cycles, but as we now show, if there are any, they must have zero length. This observation is used to help prove our unique extension result.

Lemma 7. *If the allocation function $g\colon V \to A$ is dominant strategy implementable, then any cycle of the zero 2-cycle graph Γ_g^2 has zero length.*

Proof. By Lemma 6, for any $i, j \in M$ for which $i \sim j$ in Γ_g^2, $l_{ij}^+ = l_{ij}$. Because Γ_g^+ is complete and all of its cycles have zero length, it follows that any cycle of Γ_g^2 must have zero length. □

Theorem 11 demonstrates that connectedness of the zero 2-cycle graph is sufficient for the uniqueness of a universal domain extension.[17]

Theorem 11. *If the allocation function $g\colon V \to A$ is dominant strategy implementable and the zero 2-cycle graph Γ_g^2 is connected, then g has a unique universal domain extension $g^+\colon \mathbb{R}^m \to A$.*

Proof. Consider any three nodes $i, j, k \in M$ of Γ_g^2 for which $i \sim j$ and $j \sim k$, but $i \not\sim k$. By Lemma 7, the length of the path from node i to node k via node j is the negative of the reverse path. Adding the arc from node k to node i to the first path results in a cycle. Moreover, there is a unique arc length l_{ki}^* that results in this cycle having zero length. The reverse cycle only

[17]Recall that the allocation function satisfies the revenue equivalence property if the zero 2-cycle graph is connected.

has zero length if the arc from node i to node k has length $-l^*_{ki}$. The graph Γ^2_g is connected, and so by assigning lengths in this way, we have uniquely extended Γ^2_g to a graph for which all three cycles exist and have zero length. A simple induction argument shows that this way of assigning lengths to arcs that are not in Γ^2_g uniquely extends Γ^2_g to a complete graph Γ^*_g all of whose cycles have zero length. Lemmas 5 and 6 and Theorem 10 then imply that Γ^*_g and Γ^+_g coincide. Hence, there is a unique universal domain extension g^+ of g. □

9. Affine Maximizers

The universal domain extension whose existence was established in the preceding section is now used to show that any one-person allocation function that is dominant strategy implementable is an affine maximizer. More precisely, there is a piecewise affine function of the type vector that generates the allocation function by, for each type, maximizing this function over the set of alternatives.

Consider any dominant strategy implementable allocation function g and any universal domain extension g^+ of it. By analogy with the definition of $\bar{l}_i$ in (6), let

$$\bar{l}^+_i = \frac{1}{m} \sum_{j \in M} l^+_{ji}, \quad \forall i \in M, \tag{27}$$

be the average length of all the arcs in the allocation graph of g^+ that terminate at node i. These values are the parameters in the objective function in the affine maximization problem.

This objective function is constructed as follows. For each type v in the domain of the allocation function, the difference $v_i - \bar{l}^+_i$ is computed for each alternative a_i. The requisite objective function assigns the maximum value of these differences to v. Theorem 12 shows that the alternative chosen by the allocation function g maximizes this function.

Theorem 12. *If the allocation function $g\colon V \to A$ is dominant strategy implementable, then*

$$g(v) = a_i \text{ for some } i \in \arg\max_{i \in M} \{v_i - \bar{l}^+_i\}, \quad \forall v \in V, \tag{28}$$

where $\{\bar{l}^+_i\}$ are the average lengths defined in (27) for the dominant strategy implementable universal domain extension $g^+\colon \mathbb{R}^m \to A$ of g.

Proof. By Theorem 10, such a g^+ exists. Because g is the restriction of g^+ to V, it is sufficient to prove that

$$g^+(v) = a_i \text{ for some } i \in \arg\max_{i\in M}\{v_i - \bar{l}_i^+\}, \quad \forall v \in \mathbb{R}^m. \tag{29}$$

Consider any $v \in \mathbb{R}^m$. Suppose that $i^* \in \arg\max_{i\in M}\{v_i - \bar{l}_i^+\}$. That is,

$$v_{i^*} - \bar{l}_{i^*}^+ \geq v_j - \bar{l}_j^+, \quad \forall j \in M.$$

For any $j \neq i^*$, we thus have

$$e_{i^*j} \cdot v \geq \bar{l}_{i^*}^+ - \bar{l}_j^+.$$

Expanding the RHS of this expression and using the fact that $l_{i^*i^*}^+ = l_{jj}^+ = 0$ yields

$$e_{i^*j} \cdot v \geq \frac{1}{m}\left[l_{(i^*+1)i^*}^+ + \cdots + l_{(i^*-1)i^*}^+\right] - \frac{1}{m}\left[l_{(j+1)j}^+ + \cdots + l_{(j-1)j}^+\right]$$

Because g^+ satisfies the zero 2-cycle condition, it then follows that

$$\begin{aligned} e_{i^*j} \cdot v - l_{ji^*}^+ \geq \frac{1}{m}&\left[l_{(i^*+1)i^*}^+ + \cdots + l_{(j-1)i^*}^+ + l_{(j+1)i^*}^+ + \cdots + l_{(i^*-1)i^*}^+\right] \\ &- \frac{1}{m}\left[l_{(j+1)j}^+ + \cdots + l_{(i^*-1)j}^+ + l_{(i^*+1)j}^+ + \cdots + l_{(j-1)j}^+\right] - \frac{(m-2)}{m} l_{ji^*}^+. \end{aligned}$$

Using the zero 2-cycle condition in this inequality, we obtain

$$\begin{aligned} e_{i^*j} \cdot v - l_{ji^*}^+ \geq \frac{1}{m}&\left[l_{(i^*+1)i^*}^+ + \cdots + l_{(j-1)i^*}^+ + l_{(j+1)i^*}^+ + \cdots + l_{(i^*-1)i^*}^+\right. \\ &\left. + l_{j(j+1)}^+ + \cdots + l_{j(i^*-1)}^+ + l_{j(i^*+1)}^+ + \cdots + l_{j(j-1)}^+ + (m-2) l_{i^*j}^+\right]. \end{aligned}$$

In this inequality, the expression in square brackets consists of $(m-2)$ 3-cycles of the form $\{l_{qi^*}^+, l_{i^*j}^+, l_{jq}^+\}$. Because g^+ satisfies the zero 3-cycle condition, all of these 3-cycle have zero length. Thus,

$$e_{i^*j} \cdot v - l_{ji^*}^+ \geq 0, \quad \forall j \neq i^*.$$

Therefore, $v \in P_{i^*}$ for any $i^* \in \arg\max_{i\in M}\{v_i - \bar{l}_i^+\}$. Hence, by Theorem 2, (29) holds. □

If the zero 2-cycle condition is satisfied, then the lengths in the allocation graph Γ_g^+ for the extension g^+ are the same as the lengths in the allocation graph Γ_g for the original allocation function g. As a consequence, the parameter $\bar{l}_i^+$ in the objective function (27) is simply $\bar{l}_i$, the average length of the arcs that terminate at node i in the allocation graph for g.

Corollary 1. *If the allocation function $g\colon V \to A$ is dominant strategy implementable and satisfies the zero 2-cycle condition, then*

$$g(v) = a_i \textit{ for some } i \in \arg\max_{i \in M}\{v_i - \bar{l}_i\}, \quad \forall v \in V. \tag{30}$$

Proof. This result follows directly from Lemma 6 and Theorems 9, 10, and 12. □

Note that Corollary 1 applies when the assumptions of either Theorem 6 or Theorem 7 are satisfied.

For an n-person mechanism, let $N = \{1, \dots, n\}$ be the set of individuals and V^i be the type space of the ith individual with typical element v^i. A profile of types is a vector $\mathbf{v} = (v^1, \dots, v^n)$. An allocation function is a function $G\colon \prod_{i \in N} V^i \to A$. G is *dominant strategy implementable* if each one-person allocation function obtained by fixing the types of all but one individual is dominant strategy implementable. G is an *affine maximizer* if there exist n nonnegative scalars $w_1, \dots, w_n$ that are not all equal to zero and a scalar K_i for each $i \in M$ such that

$$G(\mathbf{v}) = a_i \text{ for some } i \in \arg\max_{i \in M}\left[\sum_{j=1}^{n} w_j v_i^j + K_i\right], \quad \forall \mathbf{v} \in \prod_{i \in N} V^i. \tag{31}$$

Roberts' Theorem (Roberts, 1979) shows that if G is dominant strategy implementable and surjective, then G is an affine maximizer if A contains at least three alternatives and each type space V^i is unrestricted.[18] Roberts' Theorem has only been shown to hold for very particular restricted type spaces (see, e.g., Carbajal et al., 2013; Mishra and Sen, 2012). For a domain for which Roberts' Theorem does apply, by considering an individual j^* for

[18]Carbajal et al. (2013) and Mishra and Sen (2012) have identified sufficient conditions for an affine maximizer to be dominant strategy implementable.

whom the weight in (31) is positive and normalizing the weights so that $w_{j^*} = 1$, the maximands in (31) can be rewritten as

$$v_i^{j^*} + \sum_{j \neq j^*} w_j v_i^j + K_i, \quad \forall i \in M. \tag{32}$$

Now fix the types of all individuals $j \neq j^*$ by setting $v^j = \bar{v}^j$. For the resulting one-person allocation function g for individual j^*, Theorem 12 and (32) imply that

$$-\bar{l}_i^+ = \sum_{j \neq j^*} w_j \bar{v}_i^j + K_i, \quad \forall i \in M.^{19}$$

Thus, the parameters in the objective function of our affine maximization problem can be equivalently expressed in terms of the variables and parameters that appear in the objective function in Roberts' Theorem for any domain for which this theorem applies, thereby demonstrating the relevance of arc lengths for Roberts' objective function.

Crowell and Tran (2016) have used tropical geometry to analyze dominant strategy incentive compatible mechanisms. The max-plus algebra of tropical geometry can be employed to provide an alternative perspective on our affine maximization results and on the geometry of the difference sets that underly them.[20] The *max-plus algebra* $(\mathbb{R}, \oplus, \odot)$ is defined with tropical addition $a \oplus b = \max\{a, b\}$ and tropical multiplication $a \odot b = a + b$. The maximand in (28) can be rewritten as the tropical polynomial

$$\left(-\bar{l}_1^+ \odot (v_1)^1 (v_2)^0 \cdots (v_m)^0\right) \oplus \cdots \oplus \left(-\bar{l}_1^+ \odot (v_1)^0 \cdots (v_{m-1})^0 (v_m)^1\right) \tag{33}$$

or, equivalently, as

$$\left(-\bar{l}_1^+ \odot v_1\right) \oplus \left(-\bar{l}_2^+ \odot v_2\right) \oplus \cdots \oplus \left(-\bar{l}_m^+ \odot v_m\right). \tag{34}$$

A tropical polynomial has integer exponents. In (33), they are all either 0 or 1. For all $v \in V$, $g(v) = a_i$ only if the ith term in the tropical sum in (34) is maximal. In other words, our problem of finding the maximum of a finite number of affine equations is equivalent to finding the largest coefficient in the corresponding tropical polynomial.

---

[19] Our assumption that $g(V)$ contains at least two alternatives ensures that j^* is in fact someone with positive weight in (31).

[20] See Maclagan and Sturmfels (2015) for an introduction to tropical geometry. Its use in Economics has been pioneered by Baldwin and Klemperer (2016).

For the dominant strategy implementable allocation function $g\colon V \to A$, let

$$T_g = \{v \in V | v \in P_i \cap P_j \text{ for distinct } i, j \in M\}. \tag{35}$$

The set T_g consists of the types that lie in a common facet of two (or more) difference sets. In tropical geometry, the set of points generated in this way from a finite set of polyhedra whose facets have normals with integer components is called a *tropical hypersurface*. Here, the normal of a facet only has two non-zero components, one of which is equal to 1 and the other is equal to -1. It is this special geometric structure that underlies our results. An implication of Theorem 12 is that T_g is the set of roots of the tropical polynomial in (34); that is, it is the set of types for which more than one coefficient in (34) has the same value.

10. Economic Applications

In this section, we illustrate our analysis with two economic applications. In the first, we consider a Vickrey (1961) auction. In the second, we consider a variant of a combinatorial auction studied by Vohra (2011).

Example 7. With a Vickrey (1961) auction of a single unit of an indivisible object, the object is allocated to the highest bidder with ties broken arbitrarily. The winner pays the second highest bid and the loser pays nothing. Vickrey has shown that it is a dominant strategy for each person to bid his true value for the object.

Consider the auction from the perspective of a given individual. The set of alternatives is $A = \{a_1, a_2\}$, where in alternative a_1, this person gets the object, whereas in alternative a_2, he does not. Let $\bar{b} > 0$ denote the highest value for the object among the other bidders. The type space V for the one-person allocation function $g\colon V \to A$ is $\mathbb{R}_+ \times \{0\}$. That is, this person can have any nonnegative value for the object, but has a zero valuation if he does not receive it. For concreteness, suppose that a tie is broken in favor of this individual.

The type space is convex and there are only two alternatives, so by Theorem 7, the zero 2-cycle condition holds and the vertices of the normalized difference sets must be identical. We have

$$l(a_2, a_1) = \inf_{v_1 \geq \bar{b}} [v_1 - v_2] = \inf_{v_1 \geq \bar{b}} v_1 = \bar{b}$$

and

$$l(a_1, a_2) = \inf_{v_1 < \bar{b}} [v_2 - v_1] = -\sup_{v_1 < \bar{b}} v_1 = -\bar{b},$$

which confirms that the length of a 2-cycle is zero. The common vertex p of the two normalized difference sets is $(\bar{b}/\sqrt{2}, -\bar{b}/\sqrt{2})$, which is the projection of $(\bar{b}, 0)$ to $\mathbf{1}^\perp$. Thus, $\hat{P}_1 = \{v \in \mathbf{1}^\perp | v_1 \geq \bar{b}/\sqrt{2})\}$ and $\hat{P}_2 = \{v \in \mathbf{1}^\perp | v_1 \leq \bar{b}/\sqrt{2})\}$.

We now confirm that g is an affine maximizer with $g(v)$ given by (30). Here, $\bar{l}_1 = \bar{b}/2$ and $\bar{l}_2 = -\bar{b}/2$ because loops have zero length. We have

$$v_1 - \frac{\bar{b}}{2} \geq v_2 - \frac{\bar{b}}{2} \leftrightarrow v_1 \geq v_2 + \bar{b} \leftrightarrow v_1 \geq \bar{b},$$

where the fact that $v_2 = 0$ is used to establish the last equivalence. Thus, in accordance with Corollary 1, the allocation function for a Vickrey auction can be described using (30).

Example 8. Vohra (2011, pp. 48–49) considers a combinatorial auction in which up to two indivisible objects are allocated to a single individual and the value of having both objects is equal to the value of the object with the highest valuation. He shows that if the 2-cycle nonnegativity condition is satisfied, then the allocation function is dominant strategy implementable. We illustrate our analysis with a simplified version of this auction in which the individual must be allocated at least one of the two objects. With this simplification, the normalized type space is two dimensional. If, as Vohra assumes, the value of receiving no object is always zero, it is straightforward to include receiving no object as an option, but at the cost of making the discussion more complex.

The set of alternatives is $A = \{a_1, a_2, a_3\}$. The individual is allocated the first object alternative in a_1, the second object in a_2, and both objects in a_3. The type space is

$$V = \{(v_1, v_2, \max\{v_1, v_2\}) | v_1, v_2 \geq 0\}.$$

Note that this type space is not convex. We can rewrite V as the union of two two-dimensional cones:

$$V = \text{pos}\{(1,1,1), (1,0,1)\} \cup \text{pos}\{(1,1,1), (0,1,1)\},$$

where

$$\text{pos}\{z^1, z^2\} = \left\{ \sum_{i=1}^{2} \alpha_i \, z^i | \alpha_i \geq 0 \right\}, \quad \forall z^1, z^2 \in \mathbb{R}^3,$$

is the positive cone spanned by z^1 and z^2. The first and last components of the vectors in the first of these conjuncts are equal, as are the second and third components of the vectors in the second of them. It thus follows that $\hat{V}$, the projection of V onto $\mathbf{1}^\perp$, can be written as

$$\hat{V} = \text{pos}\{(1,-2,1),\mathbf{0}\} \cup \text{pos}\{(-2,1,1),\mathbf{0}\}.$$

That is, $\hat{V}$ is the union of two rays with endpoints at the origin.

The normalized difference sets $\{\hat{P}_i\}$ are two-dimensional cones. The rays defining $\hat{P}_i$ are obtained by intersecting the two-dimensional hyperplanes H_{ij} for $j \neq i$ with $\mathbf{1}^\perp$. Thus, $\hat{P}_1$ is bounded by two rays, one generated by a vector in $H_{12} \cap \mathbf{1}^\perp$ and the other generated by a vector in $H_{13} \cap \mathbf{1}^\perp$. The halfspaces H_{12} and H_{13} are orthogonal to $(1,-1,0)$ and $(1,0,-1)$, respectively, so we can take the first of these vectors to be $(1,1,-2)$ and the second to be $(1,-2,1)$. We can therefore express $\hat{P}_1$ as

$$\hat{P}_1 = p^1 + \text{pos}\{(1,1,-2),(1,-2,1)\}.$$

Similarly, we have that

$$\hat{P}_2 = p^2 + \text{pos}\{(1,1,-2),(-2,1,1)\}$$

and

$$\hat{P}_3 = p^3 + \text{pos}\{(1,-2,1),(-2,1,1)\}.$$

For each of these three normalized difference sets, the angle formed by their bounding rays is $2\pi/3$. The rays bounding $\hat{P}_3$ are parallel to the two rays that generate $\hat{V}$, whereas $\hat{P}_1$ and $\hat{P}_2$ each have only one bounding ray parallel to one of the rays that generate $\hat{V}$. Moreover, because $\hat{V}$ is closed, in view of how the length l_{ij} that is used to define the H_{ij} hyperplane in (4) is determined by the infimum operation in (2), for each of the normalized difference sets, at least one facet must be contained in one of the rays that generate $\hat{V}$.[21] It follows from these observations that in order for an allocation function to be dominant strategy implementable, every vertex p^i must lie in $\hat{V}$. There are two basic cases.

Case 1: $p^3 = \mathbf{0}$. In this case, $\hat{V}$ coincides with the boundary of $\hat{P}_3$. Then p^1 can lie anywhere on the ray through $(1,-2,1)$ and p^2 can lie anywhere on

[21]Only $\hat{P}_3$ can have both facets in $\hat{V}$, and that is only possible if $\hat{V}$ coincides with these two facets.

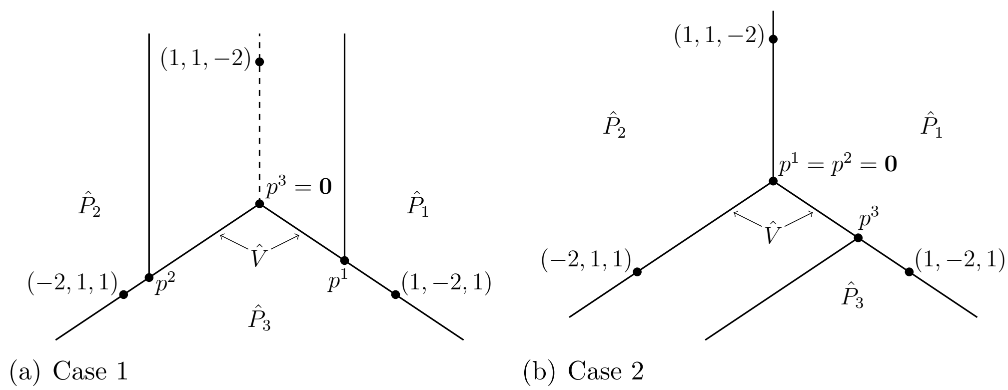

Figure 5: The Cases in Example 8.

the ray through $(-2, 1, 1)$. We have that $l_{13} + l_{31} = 0$ and $l_{23} + l_{32} = 0$, but unless all three vertices are identical, $l_{12} + l_{21} > 0$. This case is illustrated in Figure 5(a). The allocation function assigns a_3 to all types whose projection to $\hat{V}$ is between p^1 and p^2 in $\hat{V}$. It assigns either a_1 or a_3 to all types whose projection to $\hat{V}$ is weakly to the right of p^1 on the ray through $(1, -2, 1)$, with at least one of these types assigned a_1. Similarly it assigns either a_2 or a_3 to all types whose projection to $\hat{V}$ is weakly to the left of p^2 on the ray through $(-2, 1, 1)$, with at least one of these types assigned a_2.

Case 2: If $p^3 \neq \mathbf{0}$, then it must be the case that $p^1 = p^2 = \mathbf{0}$. In this case, we have that $l_{12} + l_{21} = 0$ and either $l_{13} + l_{31} = 0$ or $l_{23} + l_{32} = 0$, but not both. This case is illustrated in Figure 5(b) for the situation in which p^3 lies on the ray through $(1, -2, 1)$. The allocation function assigns either a_1 or a_2 to all types whose projection to $\hat{V}$ is the origin, a_1 to all types whose projection to $\hat{V}$ is between the origin and p^3 on the ray through $(1, -2, 1)$, a_2 to all types whose projection to $\hat{V}$ is to the left of the origin on the ray through $(-2, 1, 1)$, and either a_1 or a_3 to all types whose projection to $\hat{V}$ is weakly to the right of p^3 on the ray through $(-2, 1, 1)$, with at least one of these types assigned a_3. For the situation depicted, it is the length $l_{23} + l_{32}$ that is positive. If, however, p^3 were to lie on the ray through $(-2, 1, 1)$, then it would be the length $l_{13} + l_{31}$ that is positive.

The type space V is connected, so by Theorem 8, the zero 2-cycle graph Γ_g^2 is connected. Hence, because there are three alternatives, there can be at most one pair of nodes in the allocation graph Γ_g whose 2-cycles have positive length, which we have confirmed is the case. The unique situation in which the zero 2-cycle graph Γ_g^2 is complete occurs when $p^1 = p^2 = p^3 = \mathbf{0}$. For this reason, in both Case 1 and Case 2, the common vertex of the normalized difference sets for the extension of the allocation function to all of $\mathbb{R}^3$ is the origin. As a consequence, the lengths l_{ij}^+ are all zero. Thus, by Theorem 12, a necessary condition for the allocation function to be dominant strategy implementable is that it assigns each type a value-maximizing alternative. This is indeed the case in both Case 1 and Case 2.

11. Concluding Remarks

Gui et al. (2004), Vohra (2011), and Cuff et al. (2012), among others, have shown how the structure of dominant strategy implementable allocation functions can be identified by investigating the geometric configuration of its difference sets. We have introduced normalized difference sets and shown that it is possible to identify even more of this structure by examining their properties. In particular, we have shown that all cycles in the allocation graph having zero length is a necessary condition for an allocation function to be dominant strategy implementable if and only if the vertices of these sets coincide. While it is known from the Rockafellar–Rochet Theorem that the nonnegativity of all of the cycles in the allocation graph is necessary for dominant strategy implementability, little attention has previously been given to determining when these cycle lengths are in fact zero. Cuff et al. (2012) have shown that this is the case if the type space is the product of intervals and a mild regularity condition is satisfied. We have shown that this zero cycle condition holds in a much larger class of circumstances and that, even when not all cycles need be zero, it is often the case that many of them must be.

The necessity of a dominant strategy implementable n-person allocation function being an affine maximizer has only been established for very special type spaces (see Roberts, 1979; Carbajal et al., 2013; Mishra and Sen, 2012). We have shown that any one-person dominant strategy implementable allocation function obtained by fixing the types of the other individuals is necessarily an affine maximizer and that the objective function in this piecewise affine maximization problem has a rather simple functional form. This functional form can be identified using our finding that this one-person allocation function has a universal domain extension that satisfies the zero 2-cycle condition. This result provides a further reason as to why cycles with zero lengths are of interest. Furthermore, our affine maximizer result provides a new perspective on what dominant strategy implementability entails.

We have also contributed to the relatively small literature on dominant strategy implementability with nonconvex type spaces (see, e.g., Carbajal and Müller, 2015; Kushnir and Galichon, 2016; Mishra et al., 2014). We believe that the geometric approach used here can be fruitfully employed to further enhance our understanding of dominant strategy implementability on such domains and, more generally, of alternative concepts of implementability.

Acknowlegements

We are grateful to Juan-Carlos Carbajal, Alexey Kushnir, Debasis Mishra, Arunava Sen, and Ngoc Tran for discussions about these issues and to the participants in the Singapore Management University Workshop on Mechanism Design for their comments.

References

Archer, A., Kleinberg, R., 2014. Truthful germs are contagious: A local-to-global characterization of truthfulness. Games and Economic Behavior 86, 340–366.

Ashlagi, I., Braverman, M., Hassidim, A., Monderer, D., 2010. Monotonicity and implementation. Econometrica 78, 1749–1772.

Baldwin, E., Klemperer, P., 2016. Understanding preferences: "demand types", and the existence of equilibrium with indivisibilities, available at http://elizabeth-baldwin.me.uk/papers/demandtypes.pdf.

Berger, A., Müller, R., Naeemi, S. H., 2009. Characterizing incentive compatibility for convex valuations. In: Mavronicolas, M., Papadopoulou, V. G. (Eds.), Algorithmic Game Theory. Vol. 5814 of Lecture Notes in Computer Science. Springer-Verlag, Berlin, pp. 24–35.

Bikhchandani, S., Chatterji, S., Lavi, R., Mu'alem, A., Nisan, N., Sen, A., 2006. Weak monotonicity characterizes deterministic dominant-strategy implementation. Econometrica 74, 1109–1132.

Börgers, T., 2015. An Introduction to the Theory of Mechanism Design. Oxford University Press, New York.

Carbajal, J. C., McLennan, A., Tourky, R., 2013. Truthful implementation and preference aggregation in restricted domains. Journal of Economic Theory 148, 1074–1101.

Carbajal, J. C., Müller, R., 2015. Implementability under monotonic transformations of differences. Journal of Economic Theory 160, 114–131.

Crowell, R. A., Tran, N. M., 2016. Tropical geometry and mechanism design, available at `arxiv.org/pdf/1606.04880v2`.

Cuff, K., Hong, S., Schwartz, J. A., Wen, Q., Weymark, J. A., 2012. Dominant strategy implementation with a convex product set of valuations. Social Choice and Welfare 39, 567–597.

Gui, H., Müller, R., Vohra, R. V., 2004. Characterizing dominant strategy mechanisms with multi-dimensional types. Discussion Paper No. 1392, Center for Mathematical Studies in Economics and Management Science, Northwestern University.

Heydenreich, B., Müller, R., Uetz, M., Vohra, R. V., 2009. Characterization of revenue equivalence. Econometrica 77, 307–316.

Kushnir, A., Galichon, A., 2016. Monotonicity and implementation: Beyond convex domains, paper presented at the Vanderbilt Mechanism Design Conference.

Maclagan, D., Sturmfels, B., 2015. Introduction to Tropical Geometry. Vol. 161 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI.

Mishra, D., Pramanik, A., Roy, S., 2014. Multidimensional mechanism design in single peaked type spaces. Journal of Economic Theory 153, 103–116.

Mishra, D., Sen, A., 2012. Roberts' Theorem with neutrality: A social welfare ordering approach. Games and Economic Behavior 75, 283–298.

Myerson, R. B., 1981. Optimal auction design. Mathematics of Operations Research 6, 58–73.

Roberts, K., 1979. The characterization of implementatable choice rules. In: Laffont, J.-J. (Ed.), Aggregation and Revelation of Preferences. North-Holland, Amsterdam, pp. 321–348.

Rochet, J.-C., 1987. A necessary and sufficient condition for rationalizability in a quasi-linear context. Journal of Mathematical Economics 16, 191–200.

Rockafellar, R. T., 1970. Convex Analysis. Princeton University Press, Princeton, NJ.

Saks, M., Yu, L., 2005. Weak monotonicity suffices for truthfulness on convex domains. In: Riedl, J. (Ed.), Proceedings of the 6th ACM Conference on Electronic Commerce (EC'05). Association of Computer Machinery, New York, pp. 286–293.

Vickrey, W., 1961. Counterspeculation, auctions, and competitive sealed tenders. Journal of Finance 16, 8–37.

Vidali, A., 2009. The geometry of truthfulness. In: Leonardi, S. (Ed.), Internet and Network Economics. Vol. 5929 of Lecture Notes in Computer Science. Springer-Verlag, Berlin, pp. 340–350.

Vohra, R. V., 2011. Mechanism Design: A Linear Programming Approach. Vol. 47 of Econometric Society Monographs. Cambridge University Press, Cambridge.