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Identification of matrix-valued factor models

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Abstract

The analysis of matrix-valued time series has been popular in recent years. When the dimensions of the matrix observations are large, one can use the matrix-valued factor model to extract information from the data. However, as in standard factor analysis, the common factors and factor loadings are not separately identifiable. This note considers two sets of identification restrictions that help exactly identify the model.

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1 Introduction

Recent years have witnessed an increasing availability of matrix-valued time series. Examples include bilateral data like input-out tables and social networking data, or extended panel data consisting of multiple countries and measurements. In contrast to a typical vector-valued variable, there are two cross-sectional dimensions. Intuitively, variables in the same row or column of $\mathbf{X}_t$ usually share more commonality than those in different rows and columns. To capture this special factor structure, we consider the matrix-valued factor model, which takes the form

$$\mathbf{X}_t = \mathbf{R}\mathbf{F}_t\mathbf{C}' + \mathbf{E}_t, \qquad t = 1, \ldots, T. \tag{1}$$

MVFM is an extension to the approximate factor model, which has become one of the main workhorses in empirical finance and macroeconometrics since its introduction by Chamberlain and Rothschild (1983). Assuming all dynamics of the observed variables are captured by the factors, Wang, Liu, and Chen (2019) and Chen, Tsay, and Chen (2020) extend the factor model to incorporate matrix-valued time series. Chen, Yang, and Zhang (2022) further allow multi-dimensional observations. Assuming a strong factor structure as in Bai and Ng (2002) and Stock and Watson (2002a,b), Chen and Fan (2021) propose an estimation procedure called α-PCA and show its consistency and asymptotic normality.

However, as in standard factor models, the latent factors and factor loadings are not separately identifiable. The aforementioned estimators are only consistent up to some unknown rotations. This makes interpretation of the factors and structural analysis difficult. For this purpose, one may impose identification restrictions on the model. This note provides the theoretical justification for this. This note extends two sets of identification restriction proposed by Bai and Ng (2013) to the context of MVMF. We show that under the studied restrictions, the common factors and factor loadings can be consistently estimated without any rotations.

2 Model and estimation

This paper considers the MVFV

$$\mathbf{X}_t = \mathbf{R}\mathbf{F}_t\mathbf{C}' + \mathbf{E}_t, \qquad t = 1, \ldots, T.$$

Here $\mathbf{X}_t$ is an $N_R \times N_C$ matrix of observed data; $\mathbf{R} = (\mathbf{r}_1 \cdots \mathbf{r}_{N_R})'$ and $\mathbf{C} = (\mathbf{c}_1 \cdots \mathbf{c}_{N_C})'$ are respectively an $N_R \times K_R$ and an $N_C \times K_C$ matrix of the row and column loadings; $\mathbf{F}_t$ is a $K_R \times K_C$ matrix of the latent factors; and $\mathbf{E}_t$ is an $N_R \times N_C$ matrix of idiosyncratic error terms.

To estimate the model, Chen and Fan (2021) propose an estimation procedure called α-

PCA, which is an extension of the popular principle component analysis method. Let $\widetilde{\alpha} = \sqrt{\alpha+1} - 1$ and $\widetilde{\mathbf{X}}_t = \mathbf{X}_t + \widetilde{\alpha}\overline{\mathbf{X}}_t$, we define the image covariance matrices

$$\widetilde{\mathbf{G}}_R = (N_R N_C T)^{-1} \sum_{t=1}^{T} \widetilde{\mathbf{X}}_t \widetilde{\mathbf{X}}_t' \qquad \text{and} \qquad \widetilde{\mathbf{G}}_C = (N_R N_C T)^{-1} \sum_{t=1}^{T} \widetilde{\mathbf{X}}_t' \widetilde{\mathbf{X}}_t.$$

The row and column projection matrices are given by $\sqrt{N_R}$ times the eigenvectors corresponding to the K_R largest eigenvalues of $\widetilde{\mathbf{G}}_R$ and $\sqrt{N_C}$ times the eigenvectors corresponding to the K_C largest eigenvalues of $\widetilde{\mathbf{G}}_C$ respectively, denoted by $\widetilde{\mathbf{R}}$ and $\widetilde{\mathbf{C}}$. Finally, the matrix of common factors $\mathbf{F}_t$ is computed as

$$\widetilde{\mathbf{F}}_t = N_R^{-1} N_C^{-1} \widetilde{\mathbf{R}}' \mathbf{X}_t \widetilde{\mathbf{C}}, \qquad t = 1, \ldots, T. \tag{2}$$

To get rid of the hyperparameter, we will set $\alpha = 0$ throughout this note. This method is then equivalent to 2DSVD of Ding and Ye (2005).

3 Factor identification

As in conventional factor analysis, the common factors and factor loadings are not separately identifiable. For any invertible matrices $\mathbf{H}_R$ and $\mathbf{H}_C$,

$$\mathbf{X}_t = \mathbf{R}\mathbf{F}_t\mathbf{C}' + \mathbf{E}_t = (\mathbf{R}\mathbf{H}_R)(\mathbf{H}_R^{-1}\mathbf{F}_t\mathbf{H}_C'^{-1})(\mathbf{C}\mathbf{H}_C)' + \mathbf{E}_t = \mathbf{R}^*\mathbf{F}_t^*\mathbf{C}^{*\prime} + \mathbf{E}_t.$$

In order to identify the model, we need to impose K_C^2 and K_R^2 restrictions on the row and column factor loadings respectively. Inspired by Bai and Ng (2013), the following schemes are studied:[1]

(Res1) Partitioning $\mathbf{C} = (\mathbf{C}_1', \mathbf{C}_2')'$, we restrict $\mathbf{C}_1$ to be an invertible lower triangular matrix with positive diagonal entries. Moreover, $N_C^{-1}\mathbf{C}'\mathbf{C} = \mathbf{I}_{K_C}$.

(Res2) Partitioning $\mathbf{C} = (\mathbf{C}_1', \mathbf{C}_2')'$, we restrict $\mathbf{C}_1 = \mathbf{I}_{K_C}$.

Under the first set of identification restrictions, the first column of $\mathbf{X}_t$ is affected by the first column of $\mathbf{F}_t$ only, while the second column of $\mathbf{X}_t$ is affected by the first two columns of $\mathbf{F}_t$ only, and so on. Under the second set of the restrictions, each of the first K_C columns in $\mathbf{X}_t$ is affected by one column of $\mathbf{F}_t$ only. Two remarks are made here: First, the above restrictions are imposed on $\mathbf{R}$ and $\mathbf{C}$ independently, i.e., we can impose **Res1** on one of them and **Res2** on the other. Second, in contrast to Bai and Ng (2013), no restriction is imposed on the common factors.

[1] Here we only consider the restrictions on $\mathbf{C}$, since those on $\mathbf{R}$ can be applied analogously by taking transpose of $\mathbf{X}_t$.

Under each set of identification restrictions, we transform the estimated loadings such that the restrictions are satisfied. Let $\widetilde{\mathbf{C}} = (\widetilde{\mathbf{C}}_1', \widetilde{\mathbf{C}}_2')'$ be the partition of $\widetilde{\mathbf{C}}$ such that $\widetilde{\mathbf{C}}_1$ is square:

(Res1) We obtain the QR decomposition of $\widetilde{\mathbf{C}}_1' = \mathbf{Q}_C \mathbf{U}_C$, where $\mathbf{Q}_C$ is a $K_C \times K_C$ orthogonal matrix and $\mathbf{U}_C$ is upper triangular. The transformed column loading estimate is then $\widehat{\mathbf{C}} = \widetilde{\mathbf{C}} \mathbf{Q}_C$. The rotation matrix becomes $\mathbf{H}_C^* = \mathbf{H}_C \mathbf{Q}_C$.

(Res2) The column loading is transformed as $\widehat{\mathbf{C}} = \widetilde{\mathbf{C}} \widetilde{\mathbf{C}}_1^{-1}$. The rotation matrix becomes $\mathbf{H}_C^\dagger = \mathbf{H}_C \mathbf{C}_1^{-1}$.

Similarly, we can define $\mathbf{H}_R^* = \mathbf{H}_R \mathbf{Q}_R$ and $\mathbf{H}_R^\dagger = \mathbf{H}_R \widetilde{\mathbf{R}}_1^{-1}$, where $\mathbf{Q}_R$ is obtained from the QR decomposition of $\widetilde{\mathbf{R}}_1'$. Let $\delta_{NT,R} = \min\{\sqrt{N_C T}, N_R\}$ and $\delta_{NT,C} = \min\{\sqrt{N_R T}, N_C\}$, we show the following.

Theorem 3.1

*Under **Res1** or **Res2** and the respective transformations,* $\mathbf{H}_C^* = \mathbf{I}_{K_C} + O_p(\delta_{NT,C}^{-1})$ *and* $\mathbf{H}_C^\dagger = \mathbf{I}_{K_C} + O_p(\delta_{NT,C}^{-1})$*. Similarly,* $\mathbf{H}_R^* = \mathbf{I}_{K_R} + O_p(\delta_{NT,R}^{-1})$ *and* $\mathbf{H}_R^\dagger = \mathbf{I}_{K_R} + O_p(\delta_{NT,R}^{-1})$.

Combining with the results in Chen and Fan (2021), the transformed estimated factor loadings are now consistent without rotations. Specifically,

$$\widehat{\mathbf{r}}_i - \mathbf{r}_i = O_p\left(\delta_{NT,R}^{-1}\right), \qquad \widehat{\mathbf{c}}_j - \mathbf{c}_j = O_p\left(\delta_{NT,C}^{-1}\right).$$

For the matrix of latent factors,

$$\widehat{\mathbf{F}}_t = N_R^{-1} N_C^{-1} \widehat{\mathbf{R}}' \mathbf{X}_t \widehat{\mathbf{C}} = \mathbf{F}_t + O_p\left(\min\{\delta_{NT,C}, \delta_{NT,R}\}^{-1}\right).$$

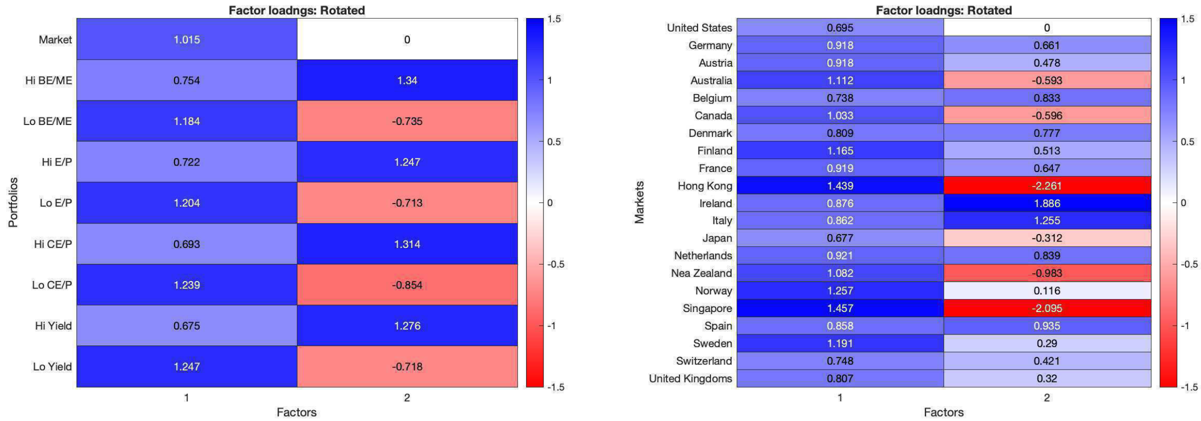
4 Application to international portfolio return data

This section applies the studied method on a set of international portfolio return data. The use of approximate factor model in asset pricing has been popular since the seminal works of Chamberlain and Rothschild (1983) and Connor and Korajczyk (1986, 1988, 1993). The linear factor model has also been used to explain international asset returns. An early review of the international asset pricing models can be found in Karolyi and Stulz (2003). More recent applications of linear factor models on international asset pricing can be found in Hou, Karolyi, and Kho (2011), Fama and French (2012, 2017), Asness, Moskowitz, and Pedersen (2013) and Amihud, Hameed, Kang, and Zhang (2015).

4.1 Data and model

We obtain the monthly international portfolio return data from the website of Kenneth R. French. The dataset covers 21 markets. For each market, we have the return data of a market

Figure 1: Estimated row and column loadings



portfolio, as well as high and low book-to-market (B/M), earnings-price (E/P), cash earnings to price (CE/P), and dividend yield (D/P) portfolios. Therefore, for each time period, the data can be arranged as

	Market	High B/M	Low B/M	...
United States	$x_{11,t}$	$x_{12,t}$	$x_{13,t}$	...
Germany	$x_{21,t}$	$x_{22,t}$	$x_{23,t}$	...
⋮				

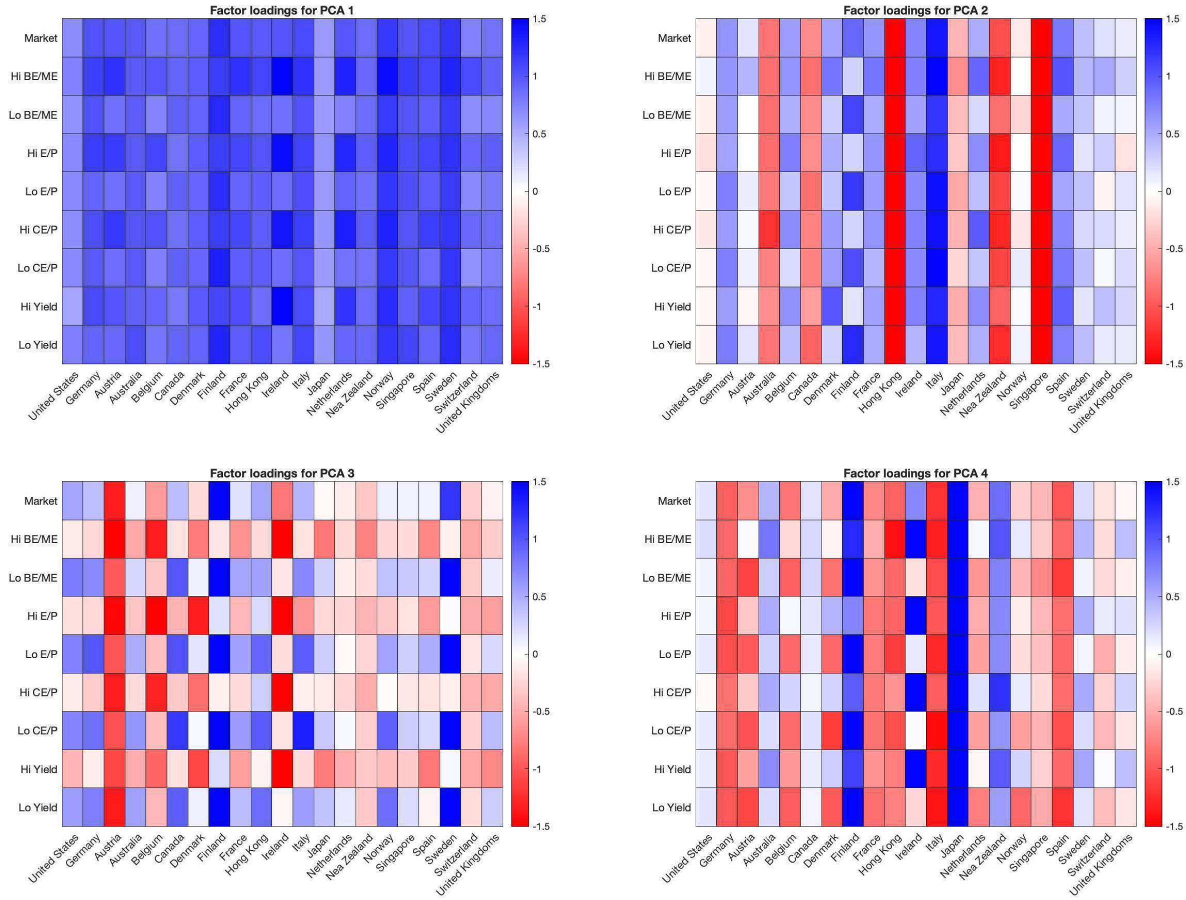
The dataset spans the period between 1991 and 2017. Due to the choice of portfolios, we will set $K_R = K_C = 2$.

4.2 Estimation and Identification

We estimate the model by 2DSVD and apply **Res1** to both the column and row loadings to identify the factors. Note that the zero restrictions are only applied to the first column and first row, i.e., we assume that the first row (United States) of $\mathbf{X}_t$ is only affected by the first row of $\mathbf{F}_t$, and that the first column (Market) of $\mathbf{X}_t$ is only affected by the first column of $\mathbf{F}_t$. The factor loadings of the remaining columns and rows are not restricted.

The factor loadings are displayed in Fig.1. We plot the column loadings on the left. Observing that all types of portfolio have a positive loadings on the first column factor, we may interpret the first column of the factor matrix as the *market factors*. For the second column, all the portfolios formed with value stocks have positive loadings, while those formed with growth stocks have negative loadings. Thus, the factors in the second column may be interpreted as the *value factors*. On the right, we observe that the loadings for the first column factor are all positive and close to one. Therefore, the first row may be interpreted as the *global factor*. In the second column, we observe that all European countries have positive loadings, while all other

Figure 2: Estimated loadings with PCA



countries have negative loadings. Moreover, we can also observe that the loadings of countries with relatively week economies, especially in the recent European debt crisis, like the GIIPS countries, are larger. We name them the *European factors*. In sum, the common factors can be interpreted as

$$\mathbf{F}_t = \begin{pmatrix} F_{\text{Global, Market}} & F_{\text{Global, Value}} \\ F_{\text{European, Market}} & F_{\text{European, Value}} \end{pmatrix}_t .$$

For comparison, we estimate a four-factor model with PCA. The estimated factor loadings are plotted in Fig.2. All portfolios have positive loadings on the first factor, regardless of their countries and the portfolio types. Therefore, we may consider it as the global market factor. For the second factor, we observe that most European countries have a positive loading, while the loadings of all non-European countries are negative or close to zero. The signs of the loading are the same across portfolio types. The second factor can be viewed as the European market factor. The third and the fourth factors are hard to interpret, though they seem to be related to the value factor. This simple exercise clearly shows the advantage of the studied method that. Compared to traditional methods, 2DSVD together with the identification restrictions can provide a better interpretation of the estimated factors.

5 Conclusion

This note studies the matrix-valued factor model. As in standard factor models, the latent factors and factor loadings are not separately identifiable. We extend the identification restrictions proposed by Bai and Ng (2013) and show that the both the common factors and factor loadings can be consistently estimated without rotations. We apply the method to a set of international portfolio return data and demonstrate how the studied method can be used to interpret the estimated factors.

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