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PEA: core-analogue for non-cohesive games

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Abstract

We present in this work the set of partition efficient anticipations, the PEA, that results from our Proposition 1, an analogue for non-balanced TU-games of the Bondareva-Shapley Theorem, as a non-empty core-analogue for non-balanced TU-games. Our Proposition 2 relates our analysis and the PEA to non-balanced super-additive market games as treated by Inoue (2012).

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1 Introduction

In this work, we are making the first steps into a widely ignored class of coalitional games, TU-games for short, namely *non-cohesive* TU-games. For those games, the fundamental concepts like *efficiency* or *feasibility* need to be modified, and the concept of (c-)core does not exist. By introducing the set of partition efficient anticipations, the PEA for short, we will be able to offer core-like solution points for those games.

Rapoport (1970) formulated two fundamental questions for the theory of TU-games:

1. "*Which coalitions are likely to form?*"
2. "*How will the members of a coalition apportion their joint payoff?*"

Most of the literature has been concerned only with the second of these two questions. By presuming that the "grand coalition" will be formed and the apportionment of the grand coalition's worth is negotiated among the players based on criteria consented to in the coalitional agreements, the first of the questions above is eliminated rather than answered. This presumption is implicit in large parts of the coalitional games literature. It is indeed optimal to form the grand coalition as long as it provides a total worth that is maximal over all partitions' of players aggregate worth. Games with this property are called *cohesive* by Osborne and Rubinstein (1994). While for cohesive games the standard notion of efficiency taking the grand coalition's worth as a benchmark is justified, it is inadequate and even defective for non-cohesive games. This fact is the main reason for our interest in an adequate solution concept that is non-empty valued on non-cohesive games. Though the core, as attractive as it may be, is only a non-feasible utopia here, we need to be content with a solution concept that is feasible in our more restricted framework, but shares some property or way of foundation with the core of balanced games.

In the context of balanced families, the players are allowed to be partial members of several coalitions simultaneously with some (for every coalition potentially different) fixed degrees of membership for each member. There, the worth of the coalition is still the classic one, but the way to deal with this is different, in particular, more flexible and more productive. The part of a coalition's worth absorbed by its members is the worth multiplied with the members' common degree of membership. The sum of the aggregate absorbed worths of the coalitions' members is too large to be feasible in a game, unless the game is c-balanced.

As we are working with *classic* TU-games where an agent's membership is indivisible Rapoport's second question "How will the members of a coalition apportion their joint payoff?" is relevant. Clearly, that question remains relevant when the membership is divisible. But for us the first question has priority even if it may finally turn out that a convincing answer can be given only by simultaneously working on both questions. Our interest in solutions for non-cohesive games is shared by Béal, S, Casajus, A et al (2021)[1] for similar reasons. They also assume Pareto efficiency, "cohesive efficiency" in their terminology, but are dealing with solutions akin to the Shapley value.

A fundamental role in our analysis is played by the choice of adequate efficiency concepts as the grand coalition cannot serve as the benchmark for Pareto optimality in non-cohesive games. Classic theory of TU-games attempts to model situations where a finite set of players cooperates in a precisely specified way in order to achieve their goals. It is presumed there that any subset of the set of players can by forming a coalition engage in a binding agreement on joint cooperation according to certain rules. We shall deviate from wide use of the technical term *coalition* in the literature in two different mutually contradictory ways: just as any subset of the set N of all players, but then also as a coalition that a subset S of N may form. We shall distinguish between all subsets of N called *syndicates* (of players) and those syndicates that become *coalitions* (of players) by signing after preplay negotiations among all players a binding contract to participate in the game (N, v) and to cooperate obeying the rules of that game. The rules are implicitly described by the coalition function v which grants any coalition a joint payoff of transferable utils ("money") called *its worth.* An example for violating the rule would be a splitting of a coalition of the game into sub-coalitions in order to receive a larger joint payoff. Yet, entering the game as coalitions would have been possible for those sub-syndicates. By distinguishing between syndicates and coalitions we follow Harsanyi (1959) and Rapoport (1970) who wrote about N: "subsets of this set can form coalitions", and adopted the term *syndicate* as a technical term for a subset of N.

We present notations, concepts, and most of our definitions in Section 2. In Section 3, we deal with the feasibility and efficiency of payoff vectors in TU-games. In Section 4, we justify for our classic context the use of partitions as opposed to balanced families on which the use of a TU-game extension is based and we define the

[1] Béal, S, Casajus, A et al (2021) quoted a first, more extended version of this paper: Aslan et al (2020), Non-cohesive TU-games: Duality and P-core. Working Paper No.2020-08, Center for International Economics, Paderborn University.

core-analogue solution concept, the PEA. Section 5 discusses the relation with coalition production economies introduced by Sun et al (2008), extended and simplified by Inoue (2012). Section 6 concludes with a summary and remarks on future research.

2 Notation and Definitions

A cooperative coalitional game with transferable utility, TU-game for short, is an ordered pair (N, v) where $N = \{1, 2, \ldots n\}$ represents the set of players and $v : 2^N \longrightarrow \mathbb{R}$ with $v(\emptyset) = 0$. The coalition function v is completely described by its restriction to $2^{\mathcal{N}}$. A *subgame* (S, v_S) of (N, v) is defined by $v_S := v|_{2^S}$.

For a non-empty finite set $N = \{1, ..., n\}$, $\mathbb{R}^N$ is an n-dimensional real vector space. For notational simplicity if $x \in \mathbb{R}^N$ and $S \subseteq N$, we write $x_S := (x_i)_{i\in S}$. We adopt a frequent convenient abuse of notation from the literature where x not only denotes a payoff vector but also the additive TU-game that it generates. Thus, $x(S) = \sum_{i\in S} x_i$. The set of all subsets of N is denoted by 2^N. We define $\mathcal{N} := 2^N \setminus \{\emptyset\}$. For any $S \subseteq N$, the indicator function $\mathbb{1}_S : N \to \{0, 1\}$ is defined by $\mathbb{1}_S(i) = 1$ if and only if $i \in S$. A *partition* π of $S \subseteq N$ is a set $\pi = \{T_1, T_2, \ldots T_m\}$ of pairwise disjoint subsets $T_i \subseteq S$ covering S. The set of partitions of $S \subseteq N$ is denoted $\Pi(S)$. A collection ß of non-empty subsets S of N is called *balanced* if there is a map $\lambda : 2^N \setminus \{\emptyset\} \to \mathbb{R}_+$ satisfying $\sum_{S\in ß} \lambda(S)\mathbb{1}_S(i) = 1$, where $\lambda(S) > 0$ for all $S \in ß$. Notice that any partition π of N is a balanced collection of subsets of N. This fact will play the central role for the foundation of the PEA as the canonical analogue for the core in non-cohesive games.

By focusing on non-cohesive games we are leaving the classes of super-additive and of balanced games, and loose access to concepts and solutions developed for these classes. However, we need to define those classes of games which are excluded from our analysis when focusing on non-cohesive games, where the grand coalition fails to be efficient.

A TU-game (N, v) is *cohesive* (or *complete*) if $v(N) = \max_{\pi\in\Pi(N)} \sum_{T\in\pi} v(T)$. The *cohesive hull* or *completion* of a TU-game (N, v) is the TU-game (N, v^c) defined by $v^c(S) = v(S)$ for all $S \subset N$ and $v^c(N) = \max_{\pi\in\Pi(N)} \sum_{T\in\pi} v(T)$. The TU-game (N, v) is super-additive if $v(S) + v(T) \leq v(S \cup T)$ for all disjoint $S, T \subseteq N$. The super-additive hull of a TU-game (N, v), denoted by $(N, \tilde{v})$, is the smallest super-additive game (N, w) such that $v(S) \leq w(S)$ for all $S \subseteq N$. Super-additive games are *totally cohesive* in the sense that all its sub-games are cohesive. For all $S \subseteq N$ holds $\tilde{v}(S) = (v_S)^c(S) = \max_{\pi\in\Pi(S)} \sum_{T\in\pi} v(T)$, and thus $\tilde{v}(N) = v^c(N) = \max_{\pi\in\Pi(N)} \sum_{T\in\pi} v(T)$. A TU-game (N, v) is *balanced* if for each balanced collection ß of subsets of N and associated system $\lambda_ß = \{\lambda(S)\}_{S\in ß}$ holds: $v(N) \geq \sum_{S\in ß} v(S)\lambda(S)$. The *balanced hull* of (N, v) is the TU-game (N, v^b) with $v^b(S) = v(S)$ for all $S \subset N$ and $v^b(N) := \max_{ß\in\mathcal{B}} \sum_{T\in ß} \lambda(T)v(T)$. A TU-game (N, v) is *totally balanced* if all subgames (S, v_S) are balanced. Neither need balanced games to be super-additive nor super-additive games to be balanced. Both are cohesive, however. For an arbitrary TU-game (N, v) holds $v(N) \leq v^c(N) = \tilde{v}(N) \leq v^b(N)$.

The concept of the core was introduced by Gillies (1959) via a domination relation for any arbitrary set P of payoff vectors for a game (N, v). Taking P as either $\{x \in \mathbb{R}^N \mid x(N) \leq v(N)\}$ or as $\{x \in \mathbb{R}^N \mid x(N) \leq v^c(N)\}$ one receives $Core(N, v)$ and $c-Core(N, v)$, respectively. The core is defined as $Core(N, v) = \{x \in \mathbb{R}^N \mid x(N) = v(N),\ x(S) \geq v(S) \text{ for all } S \subseteq N\}$, and the *c-core* of a TU-game (N, v) is the core of its cohesive hull, i.e., $c-Core(N, v) = \{x \in \mathbb{R}^N \mid x(N) = v^c(N),\ x(S) \geq v(S) \text{ for all } S \subseteq N\}$. A TU-game (N, v) is called *c-balanced* if its cohesive hull (N, v^c) is a balanced game. It is very well-known that the (c-)core of a TU-game (N, v) is non-empty if and only if (N, v) is (c-)balanced.

3 Feasibility and Efficiency

The standard version of *feasibility* and *efficiency* for TU-games is based on the realization of the grand coalition. Depending on which feasibility concept is used different types of payoff vectors are considered in the literature. Bejan and Gómez (2012) discusses three versions of feasibility defined by the following sets:

$$X(N, v) = \{x \in \mathbb{R}^N \mid x(N) \leq v(N)\}$$
$$X_\Pi(N, v) = \{x \in \mathbb{R}^N \mid x(N) \leq v^c(N)\}$$
$$X_\Lambda(N, v) = \{x \in \mathbb{R}^N \mid x(N) \leq v^b(N)\}$$

The respective subsets of *efficient* payoff vectors x are defined by replacing the inequality by equality.

As we want to get rid of the restriction by the worth of the grand coalition, we refute the first efficiency concept in the list. We also exclude the third feasibility concept that is used for balanced games, as those games are linear homogeneous game extensions. Formally, one needs to extend the notion of a game from (N, v) to a pair (N, w) with $w : 2^N \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ such that $w(S, t) := tv(S)$ for any $(S, t) \in 2^N \times \mathbb{R}_+$. Depending on the socio-economic scenario represented by (N, v), this may or may not be a coherent way of defining an extended TU-game [cf. Shapley and Shubik (1975, Footnote 5)]. Thus, we will work with the second one which is Pareto efficiency on classic coalitional TU-games and is called *cohesive efficiency* by Béal, S, Casajus, A et al (2021). It is only second best in the framework of balanced games or even totally balanced market games (cf. Shapley and Shubik (1969)).

Definition 1. *A payoff vector x for a TU-game (N, v) is called feasible if x is an element of $X_\Pi(N, v)$ and efficient if x is an element of $X^*_\Pi(N, v) := \underset{x \in X_\Pi(N,v)}{\operatorname{argmax}} \; x(N)$.*

As a consequence of our restriction to classic coalitional TU-games, we will have in our application of the *Duality Theorem of Linear Optimization* to replace the class of balanced collections which play a fundamental role in the *Bondareva-Shapley Theorem* by its subclass, the set $\Pi(N)$ of partitions of N. As we shall prove in Section 4 the worth $v^c(N)$ results from our dual linear program for non-cohesive games (N, v). There the feasibility is defined by what partitions of the player set N can possibly achieve. Their aggregate worth is maximized, rather than the worth $v(N)$ - the "benchmark" for efficiency.

4 The set $PEA(N, v)$

In their Chapter 13 on "The core", Osborne and Rubinstein (1994) wrote:

> "Throughout this chapter and the next we assume that the coalition games with transferable payoff that we study have the property that the worth of the coalition N of all players is at least as large as the sum of the worths of any partition of N. This assumption assures that it is optimal that the coalition of all players form(s),as is required by our interpretations of the solution concepts we study (though the formal analysis is meaningful without the assumption)."

This assumption makes their considered games *cohesive* as we defined it in our Section 2.

Cohesive games and even super-additive games may well have empty cores, but every TU-game with a non-empty core is necessarily cohesive. This follows from the Bondareva-Shapley Theorem that confirms the identity of balanced TU-games and TU-games with non-empty cores (cf. Myerson (1991)). In some cases a game (N, v) is not balanced, but the cohesive hull (N, v^c) may be balanced, hence (N, v^c) has then a non-empty core that is the c-core of (N, v). The game (N, v) is then c-balanced (cf. Sun et al (2008)).

We confine ourselves to analyze non-cohesive, hence non-balanced games. In particular, we want to provide a natural non-empty analogue for the the (c-)core whenever it is empty. Therefore, we shall mimic the two dual linear programs employed by Myerson (1991) in his treatment of the Bondareva-Shapley Theorem. We will do that by extending in the primal linear program the minimizing over all subsets S of N to one over all efficient partitions of N having S as a component. That decreases the minimum value. For this new primal minimization program we formulate its dual linear maximization program that necessarily leads to a decrease of the same magnitude of the maximal value as compared to Myerson's dual maximization problem.

The *Duality Theorem of Linear Optimization* asserts equality of the optimal values of the following optimization problems which are taken from Myerson (1991).

$$\text{(9.2)} \; \min_{x \in \mathbb{R}^N} x(N) \text{ subject to for all } S \in \mathcal{N}, \; x(S) \geq v(S)$$

$$\text{(9.3)} \; \max_{\mu \in \mathbb{R}^{\mathcal{N}}_+} \sum_{S \subseteq N} \mu(S) v(S) \text{ subject to for all } i \in N, \sum_{S \subseteq N : i \in S} \mu(S) = 1$$

Hence, for the respective optimizers x^* and μ^* one gets $x^*(N) = v^b(N)$. But that cannot be achieved for non-c-balanced games. An aggregate payoff $x(N) = v^b(N)$ could be possibly achieved in a non-cohesive games (N, v) only via simultaneous partial memberships of players in various coalitions. If this is the case, then the secured payoff to be allocated to each syndicate T is at least as large as its entitled worth $v(T)$.

In order to get a smaller "budget" available for the payoffs, we guarantee only the worth of those syndicates who take part in the game (N, v) as coalitions. In other words, in order to reach a smaller value as the optimal value, we first need to weaken the restrictions in (9.2) that $x(S) \geq v(S)$ holds for all syndicates S.

Then for the dual problem the maximum value has to become smaller, too, so that we have to restrict the class of balanced families of weights.

We introduce now for an arbitrary game (N,v) special subsets of the set $\Pi(N)$ of partitions: the set of efficient partitions of N is $\Pi^*(N) := \underset{\pi\in\Pi(N)}{\operatorname{argmax}}\, v(\pi)$ where $v(\pi) := \sum_{T\in\pi} v(T)$, the set of partitions of N containing S as a component is $\Pi_S(N)$, and the efficient partitions of N having S as a member is $\Pi^*_S(N) := \Pi^*(N)\cap\Pi_S(N)$.

Take any $\pi^*_S \in \Pi^*_S(N)$ for a coalition S with $v(\pi^*_S) = v^c(N)$. The aggregate payoff $x(N)$ should be big enough to guarantee to any coalition T in such a partition π^*_S the worth $(v_T)^c(T) = \tilde{v}(T)$. Notice that we are dealing here only with formed partitions with all components being (built) coalitions. Only **one** such partition can finally have that status of having entered the game. All other non-built partitions and all other syndicates remain disregarded. To do so (P) is designed as an analogue of (9.2) to guarantee the payoffs to the coalitions, but not to all syndicates. (D) is the dual maximization problem of (P).

$$\text{(P)} \min_{x\in\mathbb{R}^N} x(N) \text{ subject to for all } S\in\mathcal{N},\ x(N)\geq v(\pi_S)$$

$$\text{(D)} \max_{y\in\mathbb{R}^{\mathcal{N}}_+} \sum_{S\subseteq N} y(S)v(\pi_S) \text{ subject to } \sum_{S\subseteq N} y(S) = 1$$

Let x^* and y^* be optimizers of these dual problems, respectively. By the *Duality Theorem of Linear Optimization*, the optimal values of the dual programs are identical. The optimizers y^* in problem (D) are those probabilities y^* that put positive mass only on those S with $\pi^*_S \in \Pi^*_S(N)$. Among them are all Dirac measures $\delta_S = \mathbb{1}_{\{S\}}$ with $y^*(S) = \delta_S(S) = \mathbb{1}_{\{S\}}(S) = 1$. Thus, all partitions $\pi^*_S \in \Pi^*_S(N)$ satisfy $v(\pi^*_S) = \tilde{v}(N) = v^c(N) = x^*(N)$. This completes the proof of the following analogue of the relation between the dual programs (9.2), (9.3) and the balanced hull v^b.

Proposition 1. *The joint optimal value of the dual linear programs (P) and (D) is the worth of the grand coalition N in the cohesive hull v^c, that is $x^*(N) = v^c(N)$.*

Obviously, any optimal x^* is Pareto efficient in (N,v). We call all optimal partitions of N resulting from (P) and (D) *efficient partitions* in (N,v), i.e., a partition of N is *efficient* if its aggregate worth is $v^c(N)$. A Pareto efficient vector x is *partition efficient* if $x(S) = v(S)(=\tilde{v}(S))$ for each component S of some efficient partition of N. Although the partition efficient vectors x of (N,v) are not *aspirations*[2], they are distinguished *anticipations*.

Definition 2. *For any TU-game (N,v), the set of partition efficient anticipations is defined as*

$$PEA(N,v) = \{x\in\mathbb{R}^N \mid \exists\ \pi^*\in\Pi^*(N)\ s.t.\ x(S) = v(S)(=\tilde{v}(S))\ \forall\ S\in\pi^*\}.$$

Remark 1. It is a consequence of Proposition 1 that for any TU-game (N,v), the set $PEA(N,v)$ is non-empty.

Notice that $PEA(N,v)$ plays in non-cohesive games a completely analogous role for (N,v), $v^c(N)$, (P) and (D) to that of $Core(N,v^b)$ in cohesive games for (N,v^b), $v^b(N)$, (9.2) and (9.3).

We intentionally chose the dual linear programs (P) and (D) in order to reveal the PEA as a natural core-analogue for non-cohesive TU-games. Clearly, the renunciation of the absolute power of the grand coalition and of the flexibility of multiple coalition building has its price. We cannot just use the attractive core concept in game theoretic environments where it does not exist. A closer look at the non-cohesive TU-games shows that both are reducible to the much more transparent equivalent formulations (P_π) and (D_π):

$$(P_\pi) \min_{x\in\mathbb{R}^N} x(N) \text{ subject to for all } S\in\mathcal{N},\ x(N)\geq \tilde{v}(S)+\tilde{v}(N\setminus S)$$

$$\Leftrightarrow \min_{x\in\mathbb{R}^N} x(N) \text{ subject to } x(N)\geq v^c(N).$$

As the restriction "subject to for all $i\in N$, $\sum_{S\in\mathcal{N}} \mathbb{1}_\pi(S)\mathbb{1}_S(i) = 1$" is trivially satisfied (non-binding), we get

$$(D_\pi) \max_{\pi\in\Pi(N)} \sum_{S\in\mathcal{N}} \mathbb{1}_\pi(S)v(S) = \max_{\pi\in\Pi(N)} \sum_{S\in\pi} v(S).$$

[2] An *aspiration* is a special kind of *anticipation*, which is a payoff vector x such that for all $i\in N$, there exists some $S\subseteq N$ containing i with $x(S)\leq v(S)$. An *aspiration* $x\in\mathbb{R}^N$ for a TU-game (N,v) is an anticipation such that $x(S)\geq v(S)$ for each syndicate $S\subseteq N$ [cf. Bennett (1983)].

5 A relation to TU market games

Based on the earlier works of Shapley (1953) and Shubik (1959), *TU-market games* are introduced by Shapley and Shubik (1969) and characterized as being identical with *totally balanced* TU-games. In their follow-up on *market games*, Shapley and Shubik (1975) analysed the relation between payoff vectors in the core of a TU-game and their representation of competitive equilibria[3] in the markets inducing this game. Later on, the class of TU-market games is extended to include other games induced by or representing various market models or economies. Garratt and Qin (1996, 1997, 2000) introduced market models that can be represented by super-additive games [see also Bejan and Gómez (2017)]. Sun et al (2008) defined a coalition production economy that is induced by an arbitrary TU-game, for which the equivalence between the c-core and the utility allocation of competitive equilibria of the induced economy is established. Inoue (2012) simplified this model by reducing the number of output commodities to just one ("money") allowing two versions, one with divisible and one with indivisible labor input of agents. These two models allow or exclude, respectively, multiple jobbing of agents, i.e., allocating their available labor time to several coalitions. This gives players the opportunity to organize themselves in balanced families of syndicates rather than in partitions into various coalitions. Therefore, only the "single jobbing" model of Inoue (2012) is adequate for representing the TU-games in our framework.

While the explanation of coalition-building via competitive equilibria of a represented coalition production economy is only possible if the game is at least c-balanced [cf. Sun et al (2008)], that is still possible for arbitrary TU-games via efficiency properties. The efficient self-organization of the players into coalitions that are efficient in the production process is compatible with our framework that distinguishes syndicates from coalitions, efficient coalitions from inefficient ones, and efficient from inefficient coalition structures. We will focus now on the single jobbing model. For an elaborate interpretation and technical details, we refer to the short article by Inoue (2012).

Like in Shapley-Shubik markets, each player of the representing TU-game is endowed with one indivisible unit of an idiosyncratic good, her "labor time". Each coalition S has a technology by which it can produce $tv(S)$ for $t \geq 0$ if each member of S provides t percent of its input good labor. Indivisibility implies for each player, t to be either 0 or 1. So, the coalition is built if its production set is activated by all members of S providing their full endowments.

This is also the case if the considered game is a market game. As Shapley and Shubik (1975) remarked, the competitive payoff vectors of (N, v) are exactly those maximizing the players' aggregate utility that equals $v^c(N) = \tilde{v}(N)$. If the "right" coalitions build, they form a partition π of N that produces in total the amount $\sum_{T\in\pi} v(T) = v^c(N) = \tilde{v}(N)$. For any TU-game (N, v), there exists such an economy $\mathcal{E}_v$[4],[5]

By Inoue (2012), we know that any TU-game can be generated by $\mathcal{E}_v$, but we do not know what we can say beyond the efficiency if the game is not a c-balanced game. Proposition 2 gives a clue about efficiency in non-c-balanced games, whose straightforward proof is left to the reader.

Proposition 2. *The maximal output of $\mathcal{E}_v$ can be produced only by any vector $x \in PEA(N, v)$ and equals $x(N) = v^c(N) = \tilde{v}(N)$. All active firms are components of the same efficient partition of N underlying that $x \in PEA(N, v)$.*

6 Concluding Remarks

Our analysis of classic TU-games in this paper is based on Pareto efficiency with the aggregate worth of the most productive coalition structures as a benchmark for feasibility. Thereby, we deviated from the in large parts of the TU-literature prevalent "second best" efficiency concept based on the restriction that the grand coalition's worth is maximal among all coalitions' aggregate worths. However, these two concepts coincide on the class of cohesive games. Still, our more general concept does not satisfactorily cover **all** games in **all** of their aspects and interpretations as we have focused like most of the TU-literature on those class of coalitional TU-games where payoffs are interpreted as utilities ("profit games") rather than as dis-utilities ("loss games").

This distinction is sometimes confused with that between games with positive versus those with negative payoffs. The definition of the anti-core of a TU-game in Oishi, T, Nakayama, M et al (2016), for instance, by just reverting the inequalities in the definition of the core, contrasts strongly with Maschler et al (2013) who

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[3]A *competitive equilibrium* (or *Walrasian equilibrium*) is a pair (p, X), where p is a vector of commodity prices and X a matrix whose columns represent commodity bundles in the agents' excess demand sets at p such that markets clear, i.e., $pX = 0$.

[4]A *coalition production economy* $\mathcal{E}$ with n agents is described by a commodity space, the agents' characteristics and the coalitions' production sets satisfying some desired properties. We denote the economy generating the TU-game (N, v) by $\mathcal{E}_v$. For the details please see Inoue (2012).

[5]Notice that several coalition structures $\pi \in \Pi(N)$ may be able to produce the aggregate amount $v^c(N)$.

denote in cost games that anti-core also as core and revert the inequality in the definition of super-additive for cost-games while still calling it super-additive.

The signs of positive- or negative-valued games can always be reverted by transitions to suitable strategically equivalent games. And technically maximization or minimization problems are anyway mathematically equivalent and only depend on the sign conventions. Nevertheless, a convention may be chosen in order to represent more obviously certain features of the underlying socio-economic environment modeled by the TU-game.

As to further future research, therefore, we think of a more comprehensive analysis of TU-games, including goods and bads or allowing simultaneous treatment of pairs of dual games. That would require an adequate more general efficiency concept. A very promising candidate under this aspect could be a modification of the excess Pareto optimality introduced by Derks et al (2014) though in a cohesive games framework.

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