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On the expenditure function and welfare in random utility models

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Abstract

We extend the results on the cumulative distribution function and the expectation of the expenditure function in additive random utility models to cases of imperfect before-after correlation of the random terms, expenditure unrestricted in sign, and changing choice set. The results are of practical interest for welfare analysis.

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1 Introduction

The expenditure function in additive random utility models (ARUM[1]) was introduced by Karlström and Morey (2004). Later, Dagsvik and Karlström (2005) provided a comprehensive treatment of duality in ARUMs. The expenditure function represents the minimum income that allows the individual to retain a given utility when the price and quality of the alternatives change.

The expenditure function is a useful construct to derive welfare measures. This is because the expectation of the compensating variation, which is commonly used as welfare measure, is obtained simply from the expectation of the expenditure function.

The formula which provides the expectation of the expenditure function is an integral which is particularly convenient when choice probabilities are in closed form. When this occurs, as an example in the widely used multinomial and nested logit, the formula is a one-dimensional integral which is solved using commercially available mathematics software.

The formula of the expectation of the expenditure function is derived under the following assumptions: (i) the random terms do not change between the state before the change in price and quality and the state after, (ii) the expenditure is restricted to be positive, (iii) the choice set is unchanged between the two states. The assumptions are motivated essentially by tractability reasons.

The assumption of unchanged random terms, i.e. perfect before-after correlation, is common in the literature on welfare in random utility models. The compensating variation is computed under this assumption since the fundamental contribution by McFadden (1999). However, it is justified to consider other before-after correlation patterns of the random terms to take into account changes in unobserved attributes or intra-personal taste variation. Only recently authors have dealt with the measurement of the compensating variation under imperfect before-after correlation. Zhao et al. (2102) have carried out a numerical investigation. Delle Site and Salucci (2013) have provided both theoretical and numerical results, in particular have proved that the unconditional (with respect to the choice made) expectation of the compensating variation is independent of the before-after correlation if there is no income effect, and that the before-after correlation has impacts on the conditional expectations of the compensating variation. To date, no author has considered the assumption of imperfect before-after correlation for the expenditure function.

The restriction in sign is unnecessary. The individual, depending on her random terms, may need to compensate to an extent to which she runs into debt: in such occurrence the expenditure function takes a negative value.

Changing choice set is of interest in applications because of the possibility of deleted and new alternatives. As an example, in transportation, ARUMs are frequenly used to represent

[1]ARUM: additive random utility model

the choice for new modes, e.g. a metro or a high-speed rail, or new routes.

The present note provides the expression of the expectation of the expenditure function in cases of imperfect before-after correlation of the random terms, unrestricted-in-sign expenditure, and changing choice set.

The note is organised as follows. Section 2 provides the preliminaries from probability theory and statistics that are necessary for derivation of the subsequent results. Section 3 provides the fundamentals of random utility models and introduces the joint before-after distribution of the random terms. Section 4 deals with the expenditure function and welfare. Section 5 concludes.

2 Preliminaries

Let $\mathbf{X} = [X_1, ..., X_n]^T$, $\mathbf{X} \in \mathbb{R}^n$, be a n-variate random vector in the n-dimensional Euclidean space $\mathbb{R}^n$. An event is characterised by draws of $\mathbf{X}$ that belong to a particular set S. Let $\mathbb{P}(S)$ be the probability distribution, or law, of $\mathbf{X}$.

We make the following assumption: the probability distribution $\mathbb{P}(S)$ of $\mathbf{X}$[2] is characterised by a cumulative distribution function (CDF) denoted by $F(x_1, ..., x_n)$ and by a probability density function (PDF) denoted by $f(x_1, ..., x_n)$. Measure theory applied to probability (Athreya and Lahiri, 2010; Corbae et al., 2009) ensures the existence of a PDF if and only if the probability distribution $\mathbb{P}(S)$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^n$. This means that any (Borel-measurable) set of measure zero according to Lebesgue measure is assigned measure zero under $\mathbb{P}$. This does not ensure, however, that the PDF is continuous. We make the additional assumptions that the PDF $f(x_1, ..., x_n)$ is continuous in $\mathbb{R}^n$.

By definition of CDF we have:

$$F(x_1, ..., x_n) = \mathbb{P}(X_1 < x_1, ..., X_n < x_n)$$

$$= \int_{X_1=-\infty}^{x_1} ... \int_{X_n=-\infty}^{x_n} f(X_1, ..., X_n)\, dX_n ... dX_1 \tag{1}$$

where the multiple integral in the right-hand side is written as an iterated integral with the first integration carried out with respect to X_n and the last with respect to X_1. The Fubini's theorem[3] allows to exchange the order of integration.

[2]Random variables are denoted by capital letters, specific values by lower case letters.

[3]The Fubini's theorem states that the double integral over a rectangle of a continuous function can be computed as an iterated integral and it is possible to exchange the order of integration (Stewart, 2007). The theorem can be extended to multiple integrals and infinite range integrals.

For the first-order derivatives of the CDF we have, by an application of the fundamental theorem of calculus and the Fubini's theorem:

$$\frac{\partial F(x_1,...,x_n)}{\partial x_i} = F^i(x_1,...,x_n)$$

$$= \frac{\partial}{\partial x_i}\int_{X_i=-\infty}^{x_i}(\int_{X_1=-\infty}^{x_1}...\int_{X_{i-1}=-\infty}^{x_{i-1}}\int_{X_{i+1}=-\infty}^{x_{i+1}}...\int_{X_n=-\infty}^{x_n} f(X_1,...,X_n)$$

$$dX_n...dX_{i+1}dX_{i-1}...dX_1)dX_i$$

$$= \int_{X_1=-\infty}^{x_1}...\int_{X_{i-1}=-\infty}^{x_{i-1}}\int_{X_{i+1}=-\infty}^{x_{i+1}}...\int_{X_n=-\infty}^{x_n} f(X_1,...,X_{i-1},x_i,X_{i+1},...,X_n)$$

$$dX_n...dX_{i+1}dX_{i-1}...dX_1$$

$$i = 1,...n \tag{2}$$

The probability of the event $S = \{X_j < x_j\ \forall j \neq i;\ X_i \in A \subseteq \mathbb{R}\}$ can be expressed in terms of the first-order derivative of the CDF:

$$\mathbb{P}(S) = \int_A F^i(x_1,...,x_{i-1},X_i,x_{i+1},...,x_n)\,dX_i \tag{3}$$

Eqn (3) follows from eqn (2). The integrand is the first-order derivative of the CDF with respect to the i-th argument calculated in the point $[x_1,...,x_{i-1},X_i,x_{i+1},...,x_n]^T$.

The following lemma provides the expectation of a random variable of any sign in terms of its CDF.

Lemma 1. *The expectation $\mathbb{E}[X]$ of a random variable X characterised by a CDF $F(x)$ is given by:*

$$\mathbb{E}[X] = \int_0^{+\infty}[1-F(x)]\,dx - \int_{-\infty}^0 F(x)\,dx \tag{4}$$

Proof. The lemma is a well-known result in probability theory. A proof is in Delle Site and Salucci (2013). □

3 Random utility models

The standard microeconomic foundation to discrete choice models (found in McFadden, 1981) extends the classical consumer's model of microeconomics and considers an individual, endowed with income y, who consumes a discrete good and a composite good. The discrete good includes a set of J mutually exclusive alternatives. Given the utility-maximising behaviour subject to a constraint on income spent, when alternative i is chosen the individual will be characterised by a conditional indirect utility function u_i.

The utility u_i is expressed by the additively separable structure: $u_i = v_i + \epsilon_i$, i=1,..J, where v_i is the deterministic component, referred to as systematic utility, and ϵ_i is the random, or error, component. This structure for u_i defines the class of ARUMs. The systematic utility v_i of each alternative is assumed to be a strictly increasing function $v_i\left(y\right)$ of income y, and to depend on the price and other qualitative attributes of the alternative.

In the case of absence of income effects, i.e. when income does not affect choice, the systematic utilities take a linear form in income according to a common coefficient across alternatives.

When income effects are considered, i.e. when income affects choice, the functional form of systematic utilities that is commonly used in applied work is the translog. Examples of application are Herriges and Kling (1999), Franklin (2006) and Tra (2013). Another form is the quadratic. Examples of application are Jara-Díaz and Videla (1989) and Cherchi et al. (2004).

The assumption on the probability distribution of the J-variate random vector $\boldsymbol{\epsilon} = \left[\epsilon_1, ..., \epsilon_J\right]^T$defines the ARUM[4]. We assume that the random vector $\boldsymbol{\epsilon}$ is characterized by a PDF $f\left(\eta_1, ..., \eta_J\right)$ and a CDF $F\left(\eta_1, ..., \eta_J\right)$. Let Σ be the covariance matrix of size $J \times J$.

If the distribution is multivariate normal the probit model is obtained. McFadden (1978) has introduced the generalised extreme value ARUMs, usually referred to as GEV[5], which are obtained when the distribution is multivariate extreme value with particular properties. The multinomial logit is a GEV where the random terms are independently and identically distributed (i.i.d.) according to a Gumbel distribution (extreme value type I) across alternatives. The multivariate normal assumption is the most natural, but probit models suffer from the limitation that choice probabilities need to be computed by simulation, which is computationally demanding, because closed-form expressions are not available. The assumption on the distribution of the random terms of GEV is motivated by tractability reasons, because choice probabilities have closed-form expressions.

Consider now two states of the world, the state before a change in price and quality of the alternatives, and the state after. Consider that the set of alternatives and the random

[4]Random terms are denoted by ϵ, specific values by η.

[5]GEV: generalised extreme value

terms also may change between the two states. Quantities in the before state are denoted by the prime symbol $'$, quantities in the after state by the double prime symbol $''$.

Consider the joint distribution of the vector of random terms in the before state $\boldsymbol{\epsilon'} = [\epsilon'_1, ..., \epsilon'_{J'}]$ and the vector of random terms in the after state $\boldsymbol{\epsilon''} = [\epsilon''_1, ..., \epsilon''_{J''}]$. The joint before-after distribution of the random terms is a $(J' + J'')$-dimensional multivariate: let $h\left(\eta'_1, ..., \eta'_{J'}, \eta''_1, ..., \eta''_{J''}\right)$ be its PDF, $H(\eta'_1, ..., \eta'_{J'},\ \eta''_1, ..., \eta''_{J''})$ its CDF, and Ξ its covariance matrix of size $(J' + J'') \times (J' + J'')$. Two special cases are the one where the before and after random terms of each alternative are independent and the one where the before and after random terms are the same, i.e. perfect correlation.

In welfare analysis two states are compared, the state before the change induced by a policy, and the state after. The interest is in both the change in the systematic part of the utility and the change in the random terms, because the computation of the welfare measure is based on total utility. Thus, the consideration of the joint before-after distribution of the random terms is key in welfare analysis.

The joint before-after distribution of the random terms needs to satisfy the property that the multivariate marginal distribution of the before random term vector and the multivariate marginal distribution of the after random term vector equal the distribution of the random terms for the ARUM under consideration. Notice that the marginal distributions do not uniquely determine the joint distribution, while the converse is true.

Denote by $h_{\boldsymbol{\epsilon'}}$ and $h_{\boldsymbol{\epsilon''}}$ the marginal PDFs of, respectively, the before random terms and the after random terms. We have:

$$h_{\boldsymbol{\epsilon'}}\left(\eta_1, ..., \eta_{J'}\right) = \int ... \int_{\mathbb{R}^{J''}} h\left(\eta'_1, ..., \eta'_{J'}, \eta''_1, ..., \eta''_{J''}\right) d\eta''_{J''} ... d\eta''_1 = f\left(\eta_1, ..., \eta_{J'}\right) \tag{5}$$

$$h_{\boldsymbol{\epsilon''}}\left(\eta_1, ..., \eta_{J''}\right) = \int ... \int_{\mathbb{R}^{J'}} h\left(\eta'_1, ..., \eta'_{J'}, \eta''_1, ..., \eta''_{J''}\right) d\eta'_{J'} ... d\eta'_1 = f\left(\eta_1, ..., \eta_{J''}\right) \tag{6}$$

The integral in eqn (5) marginalizes out the after random terms. The integral in eqn (6) marginalizes out the before random terms. Eqns (5) and (6) impose the equalities in terms of PDFs. It follows that the equalities are satisfied also for the marginals of the CDFs: $H_{\boldsymbol{\epsilon'}}\left(\eta_1, ..., \eta_{J'}\right) = F\left(\eta_1, ..., \eta_{J'}\right)$ and $H_{\boldsymbol{\epsilon''}}\left(\eta_1, ..., \eta_{J''}\right) = F\left(\eta_1, ..., \eta_{J''}\right)$.

4 Expenditure function and welfare

Consider the expenditure $m_i\left(U'\right)$, conditional on the choice of alternative i in the after state, necessary to achieve the utility level of the before state $U' = \underset{i=1,..J'}{max}\, u'_i = \underset{i=1,..J'}{max}\left[v'_i\left(y\right) + \epsilon'_i\right]$. The conditional expenditure $m_i\left(U'\right)$ satisfies: $v''_i\left[m_i\left(U'\right)\right] + \epsilon''_i = U'$, $i = 1, ..J''$.

The expenditure function $M(U')$ is by definition (Karlström and Morey, 2004; Dagsvik and Karlström, 2005):

$$M(U') = \min_{i=1,..J''}[m_i(U')] \tag{7}$$

The expenditure function $M(U')$ is positive if the systematic utilities have a translog form in income, because the argument of the logarithm is positive. This is not so if the systematic utilities have a linear or quadratic form. In such cases, the expenditure function can be either positive or negative.

The compensating variation C, by definition (McFadden, 1999), satisfies:

$$U' = \underset{i=1,..J''}{max}\left[v_i''\left(y-C\right)+\epsilon_i''\right] \tag{8}$$

The conditional compensating variation c_i satisfies: $U' = v_i''\left(y-c_i\right)+\epsilon_i''$, $i=1,..J''$. Due to the increasing monotonicity of the systematic utilities in income, we have:

$$C = \underset{i=1,..J''}{max}\, c_i \tag{9}$$

By definition of conditional expenditure and conditional compensating variation, we have: $m_i\left(U'\right) = y - c_i$, $i=1,..J''$. Therefore, by eqns (7) and (9) we get:

$$M\left(U'\right) = \min_{i=1,..J''}[m_i(U')] = \min_{i=1,..J''}[y-c_i] = y - \max_{i=1,..J''} c_i = y - C \tag{10}$$

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By taking expectations we get:

$$\mathbb{E}\left[M\left(U'\right)\right] = y - \mathbb{E}\left[C\right] \tag{11}$$

which establishes the relationship between the expectation of the expenditure function and the expectation of the compensating variation.

The following proposition provides the CDF $\Gamma(m)$ of the expenditure function $M(U')$ in terms of the joint before-after CDF of the random terms H. Let $H^i\left(\eta_1',...,\eta_{J'}',\eta_1'',...,\eta_{J''}''\right)$ denote the derivative of H with respect to the i-th argument η_i'.

Proposition 1. *The CDF $\Gamma(m)$ of the expenditure function $M(U')$ is given by:*

$$\Gamma(m) = 1 - \sum_{i=1,..J'} \int_{-\infty}^{+\infty} H^i(\epsilon_i' + v_i' - v_1', ..., \epsilon_i', ..., \epsilon_i' + v_i' - v_{J'}',$$

$$\epsilon_i' + v_i' - v_1''(m), ..., \epsilon_i' + v_i' - v_i''(m), ..., \epsilon_i' + v_i' - v_{J''}''(m))d\epsilon_i' \tag{12}$$

Proof. Consider the event A that the expenditure function $M(U')$ is not lower than m: $A = \{M(U') \geq m\}$. The event A is the same as the event $\left\{\epsilon''_j \leq U' - v''_j(m) \;\; j = 1,..J''\right\}$ because the systematic utilities are monotonically increasing in income.

Consider the event B_i that alternative i is chosen in the state before and the utility level U' is attained:

$$B_i = \left\{v'_j + \epsilon'_j \leq v'_i + \epsilon'_i = U' \;\; \forall j \neq i, \;\; j = 1,..J'\right\}$$

$$= \left\{\epsilon'_j \leq U' - v'_j \;\; \forall j \neq i, \;\; j = 1,..J'\right\} \tag{13}$$

Consider the event $D_i = A \cap B_i$. We can write:

$$D_i = \left\{\epsilon'_j \leq U' - v'_j \;\; \forall j \neq i, \;\; j = 1,..J'; \;\; \epsilon''_j \leq U' - v''_j(m) \;\; j = 1,..J''\right\}$$

$$= \left\{\epsilon'_j \leq \epsilon'_i + v'_i - v'_j \;\; \forall j \neq i, \;\; j = 1,..J'; \;\; \epsilon''_j \leq \epsilon'_i + v'_i - v''_j(m) \;\; j = 1,..J''\right\} \tag{14}$$

Thus, the probability of event D_i can be written, by eqn (3), in terms of the joint before-after distribution of the random terms H as:

$$\mathbb{P}(D_i) = \int_{-\infty}^{+\infty} H^i(\epsilon'_i + v'_i - v'_1, ..., \epsilon'_i, ..., \epsilon'_i + v'_i - v'_{J'},$$

$$\epsilon'_i + v'_i - v''_1(m), ..., \epsilon'_i + v'_i - v''_i(m), ..., \epsilon'_i + v'_i - v''_{J''}(m))d\epsilon'_i \tag{15}$$

By definition of event D_i, the probability $\mathbb{P}(D_i)$ provides the survival function of the expenditure function $M(U')$ conditional on the choice of alternative i before. The unconditional survival function of the expenditure function $M(U')$ is given by:

$$1 - \Gamma(m) = \sum_{i=1,..J'} \int_{-\infty}^{+\infty} H^i(\epsilon'_i + v'_i - v'_1, ..., \epsilon'_i, ..., \epsilon'_i + v'_i - v'_{J'},$$

$$\epsilon'_i + v'_i - v''_1(m), ..., \epsilon'_i + v'_i - v''_i(m), ..., \epsilon'_i + v'_i - v''_{J''}(m))d\epsilon'_i \tag{16}$$

which proves the proposition. □

The following proposition provides the expectation $\mathbb{E}[M(U')]$ of the expenditure function $M(U')$ in terms of the joint before-after PDF of the random terms h and in terms of the CDF of the expenditute function $\Gamma(m)$.

Proposition 2. *The expectation $\mathbb{E}[M(U')]$ of the expenditure function is given by:*

$$\mathbb{E}[M(U')] = \int \dots \int_{\mathbb{R}^{J'+J''}} M\left(U'\right)\left(\epsilon'_1, ..., \epsilon'_{J'}, \epsilon''_1, ..., \epsilon''_{J''}\right) h\left(\epsilon'_1, ..., \epsilon'_{J'}, \epsilon''_1, ..., \epsilon''_{J''}\right) d\epsilon'_1 ... d\epsilon'_{J'} d\epsilon''_1 ... d\epsilon''_{J''} \tag{17}$$

$$\mathbb{E}[M(U')] = \int_0^{+\infty} \left[1 - \Gamma\left(m\right)\right] dm - \int_{-\infty}^0 \Gamma\left(m\right) dm \tag{18}$$

where $M\left(U'\right)\left(\epsilon'_1, ..., \epsilon'_{J'}, \epsilon''_1, ..., \epsilon''_{J''}\right)$ denotes that $M(U')$ is function of the before and after random terms.

Proof. Eqn (17) follows from definition of expectation. Eqn (18) follows from lemma 1. $\square$

The expectation of the expenditure function can be computed using eqn (17) of proposition 2 if the joint before-after PDF of the random terms h is available. This is the case when the ARUM is probit, i.e. the random terms are multivariate normal. In a multivariate normal distribution the marginals of any order (univariate and multivariate) are again normal. To obtain the marginal distribution over a subset of random variables, one only needs to drop the irrelevant variables, i.e. the variables that are to be marginalized out, from the expectation vector and the covariance matrix.

Therefore, for a MNP a joint before-after distribution with the postulated properties has as PDF:

$$h\left(\eta'_1, ..., \eta'_J, \eta''_1, ..., \eta''_J\right)$$

$$= \frac{1}{\left(2 \cdot \pi\right)^J \cdot \left|\Xi\right|^{1/2}} \cdot \exp\left(-\frac{1}{2} \cdot \left[\eta'_1, ..., \eta'_J, \eta''_1, ..., \eta''_J\right]^T \cdot \Xi^{-1} \cdot \left[\eta'_1, ..., \eta'_J, \eta''_1, ..., \eta''_J\right]\right) \tag{19}$$

where $|\Xi|$ is the determinant of the covariance matrix Ξ, and Ξ^{-1} is its inverse.

The expectation of the expenditure function in eqn (17) can be computed using Monte Carlo integration since it is relatively easy to draw from a multivariate normal. The procedure is as follows (Scheuer and Stoller, 1962). Let $\mathbf{X}$ have a multivariate normal distribution with zero mean vector and covariance matrix Ξ. Let the lower triangular matrix L be obtained by the Cholesky decomposition $L \cdot L^T = \Xi$. Then, given the random vector $\mathbf{Z}$ with independent standard normal components, the draws of $\mathbf{X}$ are obtained using the transformation: $\mathbf{X} = L \cdot \mathbf{Z}$.

The expectation of the expenditure function can be computed using Proposition 1 and eqn (18) of proposition 2 if the joint before-after CDF of the random terms H is available. When the ARUM is a multinomial logit, i.e. the random terms are i.i.d. Gumbel, it is possible to obtain such function using the bivariate distribution proposed by Gumbel (1961) and referred to as logistic model, or, according to Kotz et al. (2000), bivariate Gumbel type B. This bivariate distribution, which has standard Gumbel marginals, has CDF given by:

$$H^* (\eta', \eta'') = \exp \left\{ - \left[\exp (-s \cdot \eta') + \exp (-s \cdot \eta'') \right]^{1/s} \right\} \quad s > 1 \tag{20}$$

The Pearson's correlation coefficient, which equals the covariance since the marginals are standard distributions, is $\rho = 1 - s^{-2}$.

The joint before-after CDF defined as:

$$H (\eta'_1, ..., \eta'_J, \eta''_1, ..., \eta''_J) = \prod_{i=1,..J} H^* (\eta'_i, \eta''_i) \tag{21}$$

possesses the postulated property that the multivariate marginal of the before random terms and the multivariate marginal of the after random terms are i.i.d. Gumbel, i.e. we have the logit model in both the before and the after state.

The covariance matrix Ξ includes only within-alternative before-after correlations, identical across alternatives and equal to ρ, and no inter-alternative correlation. This can be proved by considering the joint before-after PDF (obtained from the CDF by derivation), and then writing the expressions of the bivariate marginal distributions which are needed to compute covariances. These distributions are obtained by integrating out the remaining random terms. We have zero covariance, and correlation, when the random terms are independent, with the bivariate marginal equal to the product of the univariate marginals.

The extension to changing choice set is straightforward if it assumed that alternatives that exist in one state only are uncorrelated with all other alternatives.

The expectation of the expenditure function in eqn (18) can be computed using numerical integration.

5 Conclusions

The note has provided an extension of the theory relating to the expenditure function in additive random utility models. In addition to the theoretical interest *per se*, the results are of relevance when the expectation of the expenditure function is computed for welfare purposes.

It has been shown that it is possible to define the expenditure function without imposing the positivity restriction, and that it is possible to use the expenditure function in cases where the choice set changes from one choice to another. These results are of interest in computation and in applied work.

It has also been shown how it is possible to relax the behaviourally unsound assumption of perfect before-after correlation. In this respect, the estimation of the before-after correlation emerges as a need. A literature exists on the estimation of probit models with panel data (a review is in Train, 2009), future research might tackle the estimation of the before-after correlation with GEV, and, *in primis*, with multinomial logit.

The note adds to the prevailing paradigm that considers stochastic welfare measures. The paradigm is not without limitations. A stochastic welfare measure implies a fundamental ambiguity in terms of policy advice, because, under the interpretation that regards the random terms as individual specific, most cases would see both individuals with a positive welfare measure, i.e. winners, and individuals with a negative welfare measure, i.e. losers. Recent research (Zhao et al, 2012; Delle Site and Salucci, 2013) has made this evident. The competing paradigm of representative consumer measures which is immune from this limitation has been recently reconsidered (Delle Site, 2013).

6 References

Athreya K.B., Lahiri S.N. (2010) *Measure Theory and Probability Theory*. Springer, New York.

Cherchi E., Polak J., Hyman G. (2004) “The impact of income, tastes and substitution effects on the assessment of user benefits using discrete choice models”. *Proceedings of the European Transport Conference*, 4-6 October, 2004, Strasbourg, France.

Corbae D., Stinchcombe M.B., Zeman J. (2009) *An Introduction to Mathematical Analysis for Economic Theory and Econometrics*. Princeton University Press, Princeton.

Dagsvik J.K., Karlström A. (2005) “Compensating variation and Hicksian choice probabilities in random utility models that are nonlinear in income”. *Review of Economic Studies* **72**(250), 57-76.

Delle Site P. (2013) “Integration of choice probabilities in logit”. *Economics Letters* 120(1), 57-60.

Delle Site P., Salucci M.V. (2013) “Transition choice probabilities and welfare analysis in random utility models with imperfect before-after correlation”. *Transportation Research Part B* **58**, 215-242.

Franklin J.P. (2006) “The equity effects of roadway tolls: an application of Hicksian welfare measures with income effects”. Paper presented at the *11th International Conference on Travel Behavior Research*, Kyoto, 16-20 August 2006.

Gumbel E.J. (1961) “Bivariate logistic distributions”. *Journal of the American Statistical Association* **56**(294), 335-349.

Herriges J.A., Kling C.L. (1999) "Nonlinear income effects in random utility models". *The Review of Economics and Statistics* **81**(1), 62-72.

Jara-Díaz S.R., Videla J.I. (1989) "Detection of income effect in mode choice: theory and application". *Transportation Research Part B* **23**(6), 393-400.

Karlström A., Morey E.R. (2004) "Calculating the exact compensating variation in logit and nested-logit models with income effects: theory, intuition, implementation and application". Presented at the *Meeting of the American Economic Association*, New Orleans, January 2001.

Kotz S., Balakrishnan N, Johnson N.L. (2000) *Continuous Multivariate Distributions*, Volume 1. Wiley, New York.

McFadden D. (1978) "Modelling the choice of residential location". In: Karlqvist A., Lundqvist L., Snickars F. and Weibull J. (eds), *Spatial Interaction Theory and Planning Models*. North Holland, Amsterdam, 75-96.

McFadden D. (1981) "Econometric models of probabilistic choice". In: Manski C., McFadden D. (eds) *Structural Analysis of Discrete Data with Econometric Applications*, MIT Press, Cambridge MA, 198-272.

McFadden D. (1999) "Computing willingness-to-pay in random utility models". In: Melvin R., Moore J.C., Riezman R. (eds) *Trade Theory and Econometrics - Essays in Honor of John Chipman*, Routledge, London, 253-274.

Scheuer E.M., Stoller D.S. (1962) "On the generation of normal random vectors". *Technometrics* **4**, 278-281.

Stewart J. (2007) *Multivariable Calculus*. Sixth Edition. Thomson Brooks/Cole, Belmont, CA.

Tra C.I. (2013) "Nonlinear income effects in random utility models: revisiting the accuracy of the representative consumer approximation". *Applied Economics* **45**(1), 55-63.

Train K. (2009) *Discrete Choice Methods with Simulation*. Cambridge University Press, Cambridge UK.

Zhao Y., Kockelman K., Karlström A. (2012) "Welfare calculations in discrete choice settings: an exploratory analysis of error term correlation with finite populations". *Transport Policy* **19**(1), 76-84.