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Autoregressive conditional beta

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Abstract

The capital asset pricing model provides various predictions about equilibrium expected returns on risky assets. One key prediction is that the risk premium on a risky asset is proportional to the nondiversifiable market risk measured by the asset's beta coefficient. This paper proposes a new method for estimating and drawing inferences from a time-varying capital asset pricing model. The proposed method, which can be considered a vector autoregressive model for multiple beta coefficients, is different from existing time-varying capital asset pricing models in that the effects of an exogenous variable on an asset's beta coefficient can be unambiguously determined and the codependence between the beta coefficients of individual assets can be measured and estimated.

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1. Introduction

The capital asset pricing model (CAPM) plays a fundamental role in the modern finance theory, and its core concept is the beta coefficient. In the CAPM, the market risk of a risky asset is measured by the contribution of the asset to the overall risk of the market portfolio and it is summarized by the beta coefficient of the asset. Let r_{it} be the excess return (over the risk-free rate, e.g., the three-month T-bill rate) on the i^{th} risky asset at time t and let r_{mt} be the excess return on the market portfolio at time t. Then the beta coefficient β_i for asset i is given by

$$\beta_i = \frac{\sigma_{im}}{\sigma_m^2},$$

where $\sigma_{im} = cov(r_{it}, r_{mt})$ and $\sigma_m^2 = var(r_{mt})$.

Since the seminal work by Bollerslev et al. (1988), a number of studies have explored the notion of the time-varying CAPM both theoretically and empirically (e.g., Akdeniz et al. 2003, Black et al. 1992, Bodurtha and Mark 1991, Faff et al. 2000, Jagannathan and Wang 1996, and Koutmos et al. 1994). The existing time-varying models assume that all investors and agents make predictions about future returns conditional on available information. Given the current time $t-1$, the beta coefficient is a function of an information set at time $t-1$ (denoted by I_{t-1}):

$$\beta_i(I_{t-1}) = \frac{\sigma_{im}(I_{t-1})}{\sigma_m^2(I_{t-1})},$$

where $\sigma_{im}(I_{t-1})$ and $\sigma_m^2(I_{t-1})$ are the corresponding conditional moments, that is, $\sigma_{im}(I_{t-1}) = cov(r_{it}, r_{mt}|I_{t-1})$ and $\sigma_m^2(I_{t-1}) = var(r_{mt}|I_{t-1})$. These conditional moments can be easily estimated by some GARCH-type procedure. This raises the following question: Which variable in the information set I_{t-1} is influential on the beta coefficient and, if so how is its influence transmitted? However, answering this question is not obvious when the conditional beta is defined as above. The reason can be heuristically seen from the following expression:

$$\frac{\partial \beta_i(I_{t-1})}{\partial I_{t-1}} = \sigma_m^{-4}(I_{t-1}) \left\{ \sigma_m^2(I_{t-1}) \frac{\partial \sigma_{im}(I_{t-1})}{\partial I_{t-1}} - \sigma_{im}(I_{t-1}) \frac{\partial \sigma_m^2(I_{t-1})}{\partial I_{t-1}} \right\}.$$

It is clear from the above expression that knowing the signs of $\frac{\partial \sigma_{im}(I_{t-1})}{\partial I_{t-1}}$ and $\frac{\partial \sigma_m^2(I_{t-1})}{\partial I_{t-1}}$ is not enough to determine the sign of $\frac{\partial \beta_i(I_{t-1})}{\partial I_{t-1}}$. This paper proposes a new method for modeling the beta coefficient as a vector autoregressive (VAR) process in which (i) there is no need to consider $\sigma_{im}(I_{t-1})$ and $\sigma_m^2(I_{t-1})$ separately, (ii) the effects of an exogenous variable in the information set on the beta coefficient can be determined unambiguously, and (iii) a variant of the Granger-causality test can be implemented to check for the codependence between individual assets' beta coefficients.

2. The Autoregressive Model

We consider N risky assets and the market portfolio and assume that the rate of return on those assets is collected in a $(N+1) \times 1$ vector $z_t = (r_t', r_{mt})'$ with $r_t = (r_{1t}, r_{2t}, ..., r_{Nt})'$. All available information at time t is collected in I_t. In addition, we assume that the distribution of z_t conditional on I_{t-1} is given by

$$\begin{aligned} z_t &= \mu_t + \epsilon_t, \\ \epsilon_t | I_{t-1} &\sim N(0, H_t), \end{aligned} \tag{1}$$

where $\mu_t = \mu_t(\delta, I_{t-1})$ and H_t are the conditional mean and variance of z_t, respectively. The normality condition is imposed on $\epsilon_t | I_{t-1}$ only for convenience and is not essential in the subsequent discussion. Following the methodology in Engle (2002), we specify H_t as follows:

$$H_t = h_t^{1/2} R_t h_t^{1/2}. \tag{2}$$

Here h_t is the conditional variance of r_{mt} from a univariate GARCH model given by

$$h_t = \omega + \sum_{i=1}^{p} \alpha_i \epsilon_{m,t-i}^2 + \sum_{i=1}^{q} \gamma_i h_{t-i}, \tag{3}$$

where ϵ_{mt} is the error term for r_{mt}, which is the last element of ϵ_t. The term R_t in (2) is the conditional covariance matrix of z_t *relative to* h_t, that is, it is given by

$$R_t = \begin{bmatrix} C_t & \beta_t \\ \beta_t' & 1 \end{bmatrix},$$

where $C_t = [C_{ij,t}]$ is the $N \times N$ conditional covariance matrix of r_t (relative to h_t) and $\beta_t = [\beta_{it}]$ is the $N \times 1$ conditional covariance vector between r_t and r_{mt} (relative to h_t). The $(N+1,\ N+1)$-th element of R_t is 1 by construction. Here the objective is obviously β_t, which is the $N \times 1$ vector of beta coefficients and is expressed as a function of the information set I_{t-1}. There are many ways to specify the functional form of R_t (thus β_t). In this paper, we consider the following GARCH(r, s) specification:

$$veca(R_t) = K + \Phi X_{t-1} + \sum_{n=1}^{r} \Delta_n veca(R_{t-n}) + \sum_{n=1}^{s} \Gamma_n veca(\dot{\epsilon}_{t-n} \dot{\epsilon}_{t-n}'), \tag{4}$$

where (i) $veca(R_t) = (vech(C_t)', \beta_t')'$, with $vech(C_t)$ being the column-stacking operator of the lower triangle of a symmetric matrix, (ii) X_{t-1} is a $k \times 1$ vector of exogenous variables that are likely to influence β_t, (iii) $\dot{\epsilon}_t = \epsilon_t / \epsilon_{m,t}$, (iv) K is a constant vector, and (v) $\Phi = [\Phi_{ij}], \Delta_n = [\Delta_{ij,n}], \Gamma_n = [\Gamma_{ij,n}]$ are the slope coefficients that need to be estimated.[1]

The GARCH process in (4) is fairly complicated in its general form. Thus, we illustrate some aspects of the model by considering a simple case with only two financial assets. When

[1] We note that the size of the constant vector K is $\frac{1}{2}N(N+3) \times 1$ and that the size of Φ is $\frac{1}{2}N(N+3) \times k$. On the other hand, the size of the square matrices Δ_n and Γ_n is $\frac{1}{2}N(N+3) \times \frac{1}{2}N(N+3)$.

$N=2$, $k=1$, and $r=s=1$, the process in (4) can be simplified as

$$\begin{bmatrix} C_{11,t} \\ C_{12,t} \\ C_{22,t} \\ \beta_{1t} \\ \beta_{2t} \end{bmatrix} = \begin{bmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \\ K_5 \end{bmatrix} + \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \\ \Phi_5 \end{bmatrix} X_{t-1} + \begin{bmatrix} \Delta_{11} & .... & \Delta_{14} & \Delta_{15} \\ : & : & : & : \\ : & : & : & : \\ \Delta_{41} & .... & \Delta_{44} & \Delta_{45} \\ \Delta_{51} & .... & \Delta_{54} & \Delta_{55} \end{bmatrix} \begin{bmatrix} C_{11,t-1} \\ C_{12,t-1} \\ C_{22,t-1} \\ \beta_{1t-1} \\ \beta_{2t-1} \end{bmatrix}$$
$$+ \begin{bmatrix} \Gamma_{11} & .... & \Gamma_{14} & \Gamma_{15} \\ : & : & : & : \\ : & : & : & : \\ \Gamma_{41} & .... & \Gamma_{44} & \Gamma_{45} \\ \Gamma_{51} & .... & \Gamma_{54} & \Gamma_{55} \end{bmatrix} \begin{bmatrix} \epsilon^2_{1,t-1}/\epsilon^2_{m,t-1} \\ \epsilon_{1,t-1}\epsilon_{2,t-1}/\epsilon^2_{m,t-1} \\ \epsilon^2_{2,t-1}/\epsilon^2_{m,t-1} \\ \epsilon_{1,t-1}\epsilon_{m,t-1}/\epsilon^2_{m,t-1} \\ \epsilon_{2,t-1}\epsilon_{m,t-1}/\epsilon^2_{m,t-1} \end{bmatrix}.$$

Using this simple case, we first consider the case of constant moments, that is, all the ARCH and GARCH terms in (3) and (4) are not present ($\alpha=\gamma=\Phi=\Delta=\Gamma=0$). Then it is straightforward to show that (i) $h_t=\omega=var(r_{mt})=\sigma^2_m$, (ii) $\beta_{1t}=K_4=\sigma_{1m}/\sigma^2_m=\beta_1$, and (iii) $\beta_{2t}=K_5=\sigma_{2m}/\sigma^2_m=\beta_2$. Therefore, the conditional beta coefficients collapse into the original unconditional beta coefficient in the static CAPM model.

We now note that possible relationships between the beta coefficient and the exogenous variables X_{t-1} can be easily investigated. For example, suppose that r_{1t} and r_{2t} in the above case represent the rate of return for the financial sector and that for the banking sector, respectively, for a developing country. Then we can regard β_{1t} and β_{2t} as market risk measures for both sectors. Suppose that we wish to estimate the effects of the country's financial liberalization on these two sectors. This issue can easily be addressed within the proposed framework. We can simply construct some indices measuring the degree of financial liberalization and collect the indices in X_{t-1}. Then the effect of financial liberalization on the financial sector can be expressed as

$$\frac{\partial\beta_{1t}}{\partial X_{t-1}} = \Phi_4,$$

and the effect on the banking sector can be similarly obtained.

The proposed method allows for what can be called "Granger-causality in market risk," which refers to the mechanism underlying the transmission of market risk between financial assets, portfolios, and sectors. In the above simple case, one may wish to know whether the banking sector's market risk can be transmitted to the financial sector, that is, whether β_{1t} depends on β_{2t-1}. In this case, the relevant null hypothesis can be formulated as $\Delta_{45}=0$.

3. Estimation and Inference

Let $\theta=(\theta'_1,\theta'_2)'$ be the vector of all parameters appearing in (1), (3), and (4), with θ_1 representing the parameters in (1) and (3) and θ_2 representing the parameters in (4). Note that for the main results in this section, we do not need to assume that $\epsilon_t|I_{t-1}$ is normally distributed. Even though $\epsilon_t|I_{t-1}$ is not necessarily normal, we can still construct the log-likelihood function as if $\epsilon_t|I_{t-1}$ is normal. In this sense, the constructed function can be considered the quasi-log-likelihood function which is given by

$$\begin{aligned}\mathcal{L}(\theta) &= -\frac{T(N+1)}{2}\ln(2\pi) - \frac{T}{2}\ln|H_t(\theta)| \\ &\quad -\frac{1}{2}\sum_{t=1}^{T}(z_t - \mu_t(\delta, I_{t-1}))' H_t(\theta)^{-1}(z_t - \mu_t(\delta, I_{t-1})),\end{aligned}$$

where $|H_t(\theta)|$ is the determinant of $H_t(\theta)$. The quasi-maximum likelihood (QML) estimator $\hat{\theta}$ is given by

$$\hat{\theta} = \arg\max \mathcal{L}(\theta).$$

Assuming the standard regularity conditions in White (1994), the QML estimator is (i) consistent for θ and (ii) normally distributed in large samples as follows:

$$T(\hat{\theta} - \theta) \xrightarrow{d} N(0, D^{-1}VD^{-1}),$$

where

$$\begin{aligned} D &= -E\left(\frac{1}{T}\frac{\partial^2 \mathcal{L}(\theta)}{\partial\theta\partial\theta'}\right), \\ V &= E\left(\frac{1}{T}\frac{\partial \mathcal{L}(\theta)}{\partial\theta}\frac{\partial \mathcal{L}(\theta)}{\partial\theta'}\right). \end{aligned}$$

If the normality condition for $\epsilon_t|I_{t-1}$ is indeed true, then $D = V$ such that the asymptotic variance of the QML estimator simplifies to D^{-1}, the inverse of the Fisher's information matrix. Thus, any standard inference/test procedure can be implemented using some consistent estimators $\hat{D}$, $\hat{V}$ for D, V, and an LM test statistic with h restrictions is distributed as $\chi^2(h)$. Depending on the values of N, p, q, r, and s, the dimension of θ can be large, which may make it difficult to implement the estimation procedure. In addition, inverting the matrix $H_t(\theta)$ for each time t and for each iteration can be a daunting task if N is large. Thus, we consider some simplified cases:

1. The case of no Granger-causality case: Because of some prior belief, one may assume that there is no Granger-causality in the system or that any Granger-causality is negligible. In this case, Δ_n and Γ_n are diagonal matrices, which can reduce the number of parameters substantially.

2. The bi-variate modeling approach: One may simply wish to examine the effects of some exogenous variables on the beta coefficient without considering Granger-causality in market risk. In this case, it is not necessary to estimate the entire system simultaneously. For each asset i, a simple bi-variate model can be estimated using $z_t = (r_{it}, r_{mt})'$.

3. Two-step estimation: It can be convenient to estimate the univariate GARCH(p, q) model in (3) separately. Let $\hat{\theta}_1$ be the QML estimator from the univariate GARCH(p, q) estimation. Then, conditional on the first-step estimator $\hat{\theta}_1$,the quasi-log-likelihood function is maximized to estimate θ_2 in the second stage. For a detailed justification for this two-step procedure, see Engle (2002) and Engle and Sheppard (2004). This two-step estimation procedure can be applied to both the full model and the above bi-variate model.

4. Conclusions

This paper proposes a novel model for a time-varying capital asset pricing model. The proposed model allows individual assets' beta coefficients to have a vector autoregressive structure and the effects of an exogenous variable on an asset's beta coefficient to be determined unambiguously. In addition, because of the VAR structure, a variant of the Granger-causality test can be implemented to check for the codependence between the beta coefficients of individual assets.

References

Akdeniz, L., A. Altay-Salih, and M. Caner (2003) "Time-Varying Betas Help in Asset Pricing: The Threshold CAPM" *Studies in Nonlinear Dynamics & Econometrics* 6(4), Article 1.

Black, A., P. Fraser, and D. Power (1992) "UK unit trust performance 1980-1989: a passive time-varying approach" *Journal of Banking and Fiance* 16, 1015-1033.

Bodurtha, J. and N. Mark (1991) "Testing the CAPM with time-varying risks and returns" *Journal of Finance* 46, 1485-1505.

Bollerslev, T., R.F. Engle, and J.M. Wooldridge (1988) "A capital asset pricing model with time-varying covariances" *Journal of Political Economy* 96, 116-131.

Engle, R. (2002) "Dynamic conditional correlation: a simple class of multivariate generalized autoregressive conditional heteroskedasticity models" *Journal of Business and Economic Statistics* 20, 339-350.

Engle, R.F. and K. Sheppard (2004) "Theoretical and empirical properties of dynamic conditional correlation multivariate GARCH" UCSD Discussion paper.

Faff, R.W., D. Hillier, and J. Hillier (2000) "Time varying beta risk: an analysis of alternative modelling techniques" *Journal of Business Finance and Accounting* 27, 523-554

Jagannathan, R. and Z. Wang (1996) "The conditional CAPM and the cross-section of expected returns" *Journal of Finance* 51, 3-53.

Koutmos, G., U. Lee, and P. Theodossiou (1994) "Time-varying betas and volatility persistence in international stock markets" *Journal of Economics and Business* 46, 101-112.

White, H. (1994) *Estimation, Inference and Specification Analysis.* Cambridge University Press, New York.