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Sustainability of public debt: Some theoretical considerations

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Abstract

This paper elaborates on the relationship between sustainability of public debt and the debt to GDP ratio in case the interest rate on public debt exceeds the growth rate of GDP. When the primary surplus relative to GDP positively reacts to a higher debt to GDP ratio, a bounded debt to GDP ratio guarantees sustainability. Further, an unbounded debt to GDP ratio is not compatible with sustainability, even if the primary surplus relative to GDP strictly rises as the debt to GDP ratio increases. Finally, sustainability is excluded if the initial debt to GDP ratio exceeds a critical threshold.

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1. Introduction

Modern research on sustainability of debt policies that apply statistical tests has started with the contribution by Hamilton and Flavin (1986) who analyzed whether the series of public debt in the USA contains a bubble term. Since then a great many papers have been written that try to answer the question of whether given debt policies can be considered as sustainable. The interest in that question is in part due to the fact that the latter question is not only of academic interest but that it has practical relevance, too. Hence, if tests reach the conclusion that given debt policies cannot be considered as sustainable governments should undertake corrective actions.

An important role in many of those studies on sustainability plays the interest rate, an aspect that was pointed out by Wilcox (1989) for example. Recalling that the intertemporal budget constraint of the government requires that the present value of public debt asymptotically converges to zero, the role of the interest rate that is resorted to in order to discount the stream of public debt becomes immediately clear. Therefore, tests have been conceived that reach results which are independent of the interest rate. One such test is to analyze whether public deficits inclusive of interest payments grow at most linearly, as suggested by Trehan and Walsh (1991). If that property is fulfilled a given series of public debt is sustainable because any time series that grows linearly converges to zero if it is exponentially discounted, provided the real interest rate is positive. Denoting by B public debt and by r the interest rate, another test proposed by Trehan and Walsh (1991) is to analyze whether a quasi-difference of public debt, $B_t - \lambda B_{t-1}$ with $0 \leq \lambda < 1 + r$, is stationary and whether public debt and primary surpluses are cointegrated. If government debt is quasi-difference stationary and public debt are primary surpluses are conitegrated, fiscal policy is sustainable. Hence, these two tests present alternatives where the outcome is independent of the exact numerical value of the interest rate. A survey of analyses that tested on sustainability of debt policies can be found in Afonso (2005).

Another test that has received great attention in the economics literature is the one proposed by Bohn (1995). There, it is suggested to test whether the primary surplus relative to GDP is a positive and at least linearly rising function of the debt to GDP ratio. If that property holds, a given public debt policy can be sustainable. This test is very plausible because it has a nice economic intuition: if governments run into debt today they have to take corrective actions in the future by increasing the primary surplus relative to GDP. Otherwise, public debt will not be sustainable. Testing real-world debt policies for that property one can indeed find evidence that countries behave like that (see e.g. Ballabriga and Martinez-Mongay, 2005, or Greiner et al., 2007, for countries of the euro area).

From a statistical point of view, a rise in primary surpluses as a response to higher government debt implies that the series of public debt relative to GDP should become a mean-reverting process. This holds because higher debt ratios lead to an increase in the

primary surplus relative to GDP, making the debt ratio decline and return to its mean. However, mean-reversion only holds if the reaction coefficient, determining how strongly the primary surplus reacts as public debt rises, is sufficiently large, as will be shown in detail in this paper.

In this note, our goal is to elaborate on that test from a theoretical point of view. In particular, we are interested in the behavior of the debt to GDP ratio when governments pursue sustainable debt policies. For example, one question we address is whether a sustainable debt policy is compatible with a rising debt to GDP ratio. Another question we study is whether sustainability can be given if the government does not react to rising debt ratios and whether there probably exists a critical initial debt ratio that makes a sustainable debt policy impossible.

The remainder is organized as follows. In section two the period and the inter-temporal budget constraints of the government are introduced. Section three analyzes under which conditions sustainability of public debt is assured and section four, finally, concludes the note.

2. Public debt and the budget constraint

We consider a real economy and we posit that the government cannot use seignorage or inflation to reduce its outstanding debt.

Starting point for the analysis of sustainability of public debt, then, is the accounting identity describing the accumulation of public debt in continuous time by the following differential equation:

$$\dot{B}(t) = r(t)B(t) - S(t), \tag{1}$$

with $B(t)$ real public debt[1] at time t, $r(t)$ the real interest rate, $S(t)$ the real government surplus exclusive of interest payments on public debt and the dot over a variable stands for the derivative with respect to time d/dt. A government is said to follow a sustainable debt policy if the present value of public debt converges to zero asymptotically, i.e. if it does not play a Ponzi game. This implies that $\lim_{t\to\infty} e^{-C_1(t)}B(t) = 0$ holds, with $C_1(t) = \int_0^t r(\mu)d\mu$ (see e.g. Blanchard and Fischer, 1989, chapter 2).

In the next section, we analyze conditions that must be fulfilled such that the inter-temporal budget constraint of the government holds and how the debt to GDP ratio evolves in that case.

3. Conditions for sustainability of public debt

Before we start our analysis we make two additional assumptions. First, we posit that the interest rate on government bonds exceeds the growth rate of GDP on average so that $\int r(\mu)d\mu > \int g(\mu)d\mu$, with g denoting the growth rate of GDP. We make this assumption

[1]Strictly speaking, B should be real public net debt.

because otherwise the inter-temporal budget constraint would not pose a problem for the government since it can grow out of debt in that case. In addition, this condition is fulfilled for countries of the euro area at least since the 1980's. Second, we neglect the case where public debt becomes negative meaning that the government would be a net lender. This is done for reasons of realism because a situation with negative public debt is of less relevance for countries in the euro are.

In our analysis of sustainable debt policies we are particularly interested under which conditions sustainability of public debt is given and in the question of whether a sustainable debt policy is compatible with a rising debt to GDP ratio. First, we analyze the situation where the primary surplus is a function of the debt to GDP ratio.

3.1 The primary surplus as a function of public debt

Assume that the government in the economy chooses the primary surplus to GDP ratio, $s(t) = S(t)/Y(t)$, such that it is a positive linear function of the debt to GDP ratio, $b(t) = B(t)/Y(t)$, and of an autonomous term that is independent of public debt, $\alpha(t)$ (cf. Bohn, 1995, or Canzoneri et al., 2001). The primary surplus ratio, then, can be written as

$$s(t) = \beta(t)\, b(t) + \alpha(t), \tag{2}$$

where $\beta(t)$ is the coefficient determining how strong the primary surplus reacts to changes in the public debt ratio and that is time-varying. It should be noted that any non-linear model can be approximated by a linear model with time-varying coefficients. Further, the approximation is good if the parameter changes smoothly (cf. Granger, 2008). Thus, the modelling in (2) can be justified and there does not seem to be the need for a more general function describing the response of the primary surplus to public debt.

The term $\alpha(t)$ is also time dependent and it is influenced by other economic variables, such as social spending or transitory government expenditures in general. As concerns $\alpha(t)$ we suppose that it is bounded by above and by below by a certain finite number that is constant over time. We should also like to point out that $\alpha(t)$ cannot be completely controlled by the government. The government can influence that parameter to a certain degree but has not complete control over it because $\alpha(t)$ is also affected by the business cycle for example that can affect temporary government outlays.

To study sustainability of public debt, we combine equation (1) and (2) yielding

$$\dot{B}(t) = (r(t) - \beta(t))\, B(t) - \alpha(t)\, Y(t). \tag{3}$$

With (3), the debt to GDP ratio evolves according to

$$\dot{b}(t) = (r(t) - \beta(t))\, b(t) - \alpha(t) - g(t)\, b(t). \tag{4}$$

With these two equations, we can derive our first result in proposition 1.

Proposition 1 *Assume that the upper bound of the primary surplus to GDP ratio is not binding. Then, a strictly positive reaction coefficient on average so that* $\int_0^t \beta(\mu)d\mu = \infty$ *holds for* $t \to \infty$ *guarantees sustainability of public debt.*

For $\int_0^t \beta(\mu)d\mu > \int_0^t (r(\mu) - g(\mu))\, d\mu$*, the debt to GDP ratio converges to a constant and it diverges for* $\int_0^t \beta(\mu)d\mu \leq \int_0^t (r(\mu) - g(\mu))\, d\mu$*, for* $t \to \infty$ *respectively.*

Proof: See Appendix.

This proposition demonstrates that a positive reaction coefficient on average is sufficient for sustainability of public debt.[2] If the reaction coefficient is strictly negative on average, the discounted value of public debt diverges to infinity. But proposition 1 also shows that a positive value of the reaction coefficient does not necessarily imply that the debt to GDP ratio remains constant or that it asymptotically converges to a constant. Only if the reaction coefficient exceeds the positive difference between the interest rate and the GDP growth rate on average, convergence can be guaranteed. Otherwise, the debt to GDP ratio diverges to infinity.

With the result of proposition 1, one could reach the conclusion that a sustainable debt policy is compatible with a continuously rising debt to GDP ratio, in case the reaction coefficient β is positive on average but smaller than the difference between the average interest rate and the average growth rate, $r - g$. However, when the government sets the primary surplus according to rule (2), that possibility is not given as proposition 2 demonstrates.

Proposition 2 *If the government pursues a sustainable debt policy and sets the primary surplus according to the rule given by* (2)*, the debt to GDP ratio remains bounded.*

Proof: Assume that $b(t) \to \infty$. According to (2) this implies $s(t) \to \infty$ which, however, is excluded because the primary surplus cannot become larger than GDP implying that $s(t) < 1$ must always hold. □

The result in proposition 2 is due to the fact that an unbounded debt to GDP ratio would require that the primary surplus to GDP ratio goes to infinity which, however, is not possible since the ratio of the primary surplus to GDP is bounded by above. This holds because the primary surplus must be financed out of GDP so that the ratio of the primary surplus to GDP must be smaller one. Consequently, when the government pursues a sustainable debt policy and raises the primary surplus relative to GDP as the debt to GDP ratio increases, the debt ratio must remain bounded in the long-run.

Hence, a situation may be observed where the debt to GDP ratio rises over a certain time period although the primary surplus positively reacts to higher public debt. Such an evolution of public debt may be compatible with a sustainable debt policy but it cannot go on forever. Sooner or later, the public debt to GDP ratio must become constant or converge to zero. Otherwise, sustainability is not given.

[2]That has already been shown in Greiner (2008) for example. We repeat it here for reasons of readability and for completeness of the presentation.

3.2 The primary surplus independent of public debt

In the last subsection it was assumed that the government sets the primary surplus according to the rule specified in equation (2). However, one could argue that governments can perform sustainable debt policies without reacting to higher public debt if they only chose the primary surplus sufficiently high, independent of public debt. Further, a situation is feasible where the government cannot react to higher debt since there is no scope for it because the primary surplus relative to GDP has already reached its upper bound. In both cases the reaction coefficient $\beta(t)$ would be zero.

In order to analyze that case we set $\beta(t) = 0$ and we denote by $m < 1$ the constant upper bound of the primary surplus to GDP ratio. In addition, we assume that the government sets the primary surplus to GDP ratio equal to that maximum value for all times, i.e. $s(t) = m$ for all t. Thus, the evolution of public debt is described by

$$\dot{B}(t) = r(t)\, B(t) - m\, Y(t) \tag{5}$$

and the debt to GDP ratio evolves according to

$$\dot{b}(t) = r(t)\, b(t) - m - g(t)\, b(t). \tag{6}$$

Given equation (5) and equation (6), we can derive proposition 3.

Proposition 3 *Assume that the initial debt to GDP ratio exceeds a threshold given by* $b_{crit} = m \int_0^\infty e^{-(C_1(\mu)-C_2(\mu))} d\mu$, *with* $C_1(\mu) = \int_0^\mu r(\nu) d\nu$, $C_2(\mu) = \int_0^\mu g(\nu) d\nu$, *then a sustainable debt policy is excluded.*

If the initial debt to GDP ratio is smaller than or equal to the critical threshold, the government can pursue a sustainable debt policy. In this case, the debt to GDP ratio becomes zero.

Proof: See Appendix.

Proposition 3 states that a sustainable debt policy cannot be pursued if the initial debt to GDP ratio is larger than a certain critical value. The critical value depends on how large the primary surplus relative to GDP can maximally become, m, and on the average difference between the interest rate and the growth rate, $r - g$. Hence, countries that do not stabilize their debt to GDP ratio but instead let it grow for a longer time period face the risk that they find themselves in a situation where they cannot react to higher debt to GDP ratios by raising their primary surplus relative to GDP. Then, it may become impossible to pursue a sustainable debt policy, independent of how large the primary surplus relative to GDP is set. In this case, the public debt to GDP ratio becomes unbounded asymptotically.

The proposition also demonstrates that the government can control public debt if it chooses the maximally possible value of the primary surplus, m, provided the initial debt to GDP ratio is not too large, i.e. if it is smaller than the critical value b_{crit}. In that case, sustainability of public debt is guaranteed and the debt to GDP ratio becomes zero.

4. Conclusion

In this paper, we have studied conditions under which governments can pursue sustainable debt policies. When the interest rate exceeds the growth rate of GDP on average, a given debt policy is sustainable if the primary surplus relative to GDP is a positive function of the debt to GDP ratio on average that exceeds the average difference between the interest rate and the GDP growth rate. If a government pursues such a policy the debt to GDP ratio converges to a constant. We could also demonstrate that a public debt to GDP ratio which rises in the long-run is not compatible with a sustainable debt policy.

If the primary surplus relative to GDP does not react to a rising debt ratio, a sustainable debt policy is excluded if the initial debt to GDP ratio exceeds a certain threshold, independent of how large the primary surplus is chosen.[3] If the initial debt to GDP ratio is smaller than the critical value it is possible that public debt remains sustainable provided the ratio of the primary surplus relative to GDP is sufficiently large.

A Proof of proposition 1

To prove that proposition we note that the evolution of public debt is given by equation (3). Integrating that equation and multiplying the resulting expression by $e^{-\int_0^t r(\tau)d\tau}$ to get present values gives,

$$e^{-C_1(t)}B(t) = e^{-C_3(t)}B(0) - Y(0)e^{-C_3(t)}\int_0^t e^{-C_1(\mu)+C_2(\mu)+C_3(\mu)}\alpha(\mu)d\mu, \quad (A.1)$$

with

$$\int_0^t r(\mu)d\mu = C_1(t), \int_0^\mu r(\nu)d\nu = C_1(\mu), \int_0^\mu g(\nu)d\nu = C_2(\mu), \int_0^\mu \beta(\nu)d\nu = C_3(\mu).$$

For $\lim_{t\to\infty} C_3(t) = \lim_{t\to\infty}\int_0^t \beta(\nu)d\nu = \infty$, the first term on the right hand side in $(A.1)$, i.e. $e^{-C_3(t)}\,B(0)$, converges to zero.

The second term on the right hand side in $(A.1)$ can be written as

$$\frac{Y(0)\int_0^t e^{-C_1(\mu)+C_2(\mu)+C_3(\mu)}\alpha(\mu)d\mu}{e^{C_3(t)}} := K_1(t).$$

Since $|\alpha| < \infty$ we can set $\alpha Y(0) = 1$. If $\int_0^\infty e^{-C_1(\mu)+C_2(\mu)+C_3(\mu)}d\mu$ remains bounded $\lim_{t\to\infty} C_3(t) = \infty$ guarantees that K_1 converges to zero. If $\lim_{t\to\infty}\int_0^\infty e^{-C_1(\mu)+C_2(\mu)+C_3(\mu)}$ $d\mu = \infty$, applying l'Hôpital gives the limit of K_1 as

$$\lim_{t\to\infty} K_1(t) = \lim_{t\to\infty}\frac{e^{-C_1(t)+C_2(t)}}{\beta(t)}\,.$$

[3]Of course, the primary surplus is bounded by above and strictly smaller than GDP.

Since $-C_1(t)+C_2(t)<0$ we can find a constant $k>0$ such that $K_1 \leq e^{-kt}/\beta(t)$. The right hand side in the former inequality does not converge to zero if $\beta(t)$ converged to zero exponentially. However, in that case $\lim_{t\to\infty}\int_0^t \beta(\mu)d\mu<\infty$ would hold. Consequently, in case that $\lim_{t\to\infty}\int_0^t \beta(\mu)d\mu=\infty$ holds, $\beta(t)$ cannot decline exponentially, and $K_1(t)$ converges to zero.

These considerations demonstrate that the inter-temporal budget constraint holds for $\lim_{t\to\infty}\int_0^t \beta(\mu)d\mu=\infty$ which means that the reaction coefficient $\beta(t)$ is positive on average.

The debt ratio is obtained from (4) as

$$b(t)=e^{(C_1(t)-C_2(t)-C_3(t))}b(0)-e^{(C_1(t)-C_2(t)-C_3(t))}\int_0^t e^{-(C_1(\mu)-C_2(\mu)-C_3(\mu))}\alpha(\mu)d\mu.$$

That expression shows that the debt ratio diverges to plus or minus infinity for $\int_0^t \beta(\mu)d\mu \leq \int_0^t (r(\mu)-g(\mu))\,d\mu$, while it converges in all other cases. □

B Proof of proposition 3

To prove that proposition we note that the present value of public debt is now obtained from (5) as

$$e^{-C_1(t)}B(t)=B(0)-mY(0)\int_0^t e^{-(C_1(\mu)-C_2(\mu))}d\mu.$$

The inter-temporal budget constraint is fulfilled for $\lim_{t\to\infty}e^{-C_1(t)}B(t)=0$ which implies $b(0)=m\int_0^\infty e^{-(C_1(\mu)-C_2(\mu))}d\mu$. If the initial debt to GDP ratio, $b(0)$, is larger than $m\int_0^\infty e^{-(C_1(\mu)-C_2(\mu))}d\mu$ sustainability of public debt is excluded.

The debt to GDP ratio is obtained from (6) as

$$b(t)=e^{(C_1(t)-C_2(t))}\left(b(0)-m\int_0^t e^{-(C_1(\mu)-C_2(\mu))}d\mu\right).$$

If the inter-temporal budget constraint holds we have $b(0)=m\int_0^\infty e^{-(C_1(\mu)-C_2(\mu))}d\mu$ and, thus, $\lim_{t\to\infty}b(t)=0$. □

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