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The Stackelberg equilibrium as a consistent conjectural equilibrium

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Abstract

We consider a static game with conjectural variations where some firms make conjectures while others do not. Two propositions are proved. We first show that there exists a continuum of conjectural variations such that the conjectural equilibrium locally coincides with the Stackelberg equilibrium (Proposition 1). Second, we define the conditions under which a conjectural equilibrium is a locally consistent equilibrium (i.e. such that conjectures are fulfilled). The concept of (local) consistency is restricted to firms making conjectures. Two conditions on consistency are featured: consistency within a cohort and consistency among cohorts. The Stackelberg equilibrium fulfills only the latter condition (Proposition 2). An example is provided.

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1 Introduction

The Stackelberg (1934) model is a sequential model embodying two firms, one leader and one follower. They both have perfect information about the market demand function. The leader firm moves first and makes its decision taking into account the reaction of the follower, which is correctly perceived (Negishi and Okuguchi (1972)). The follower rationally sets its own output level according to any quantity set by the leader, with the expectation that the leader will not counter-react. Similarly, the leader may expect the follower to conform to his best strategy. In a Stackelberg equilibrium, beliefs are self-fulfilled, and are therefore an essential feature of such a kind of economies.

In this note, we provide a conjectural interpretation of the Stackelberg equilibrium concept. We consider conjectural variations in a static deterministic model with many firms. These conjectures capture the way a firm anticipates the reactions of its rivals when it decides to increase its supply by one unit, in a simultaneous game (Bowley (1924), Friedman and Mezzetti (2002), Figuières et al. (2004)). In our model, we assume an asymmetry in the formation of beliefs: some firms (type 1) make conjectures while others (type 2) do not. The asymmetry could be justified for instance by some differences on costs that would put firms in asymmetric positions, and thus cause them to formulate some asymmetric conjectures.[1] But, the foundations of such an asymmetry is out of scope in this paper. The distribution of firms between each group is assumed to be exogenous. In our model, no sequential structure is assumed regarding the timing of decisions.

In the conjectural variations literature, significant research has been devoted to local consistency of static conjectures (Figuières et al. (2004)). In our framework, a locally consistent equilibrium is such that each type 1 firm is able to correctly perceive the slopes of the reaction functions of both type 2 firms and their competitors. This means that conjectures coincide with the true values of the slopes of reaction functions (Bresnahan (1981), Perry (1982), Ulph (1983)). We consider symmetric (within a cohort) constant conjectural variations (Bowley (1924), Figuières et al. (2004), Perry (1982)).[2] It is known that under constant marginal costs, the competitive equilibrium turns out to be the only consistent equilibrium (Bresnahan (1981), Perry (1982)). The existence of an asymmetry in the formation of beliefs complicates the modelling of conjectures since it introduces indirect effects. Hence, the preceding result still holds in our model.

Within the same economy, we also study a sequential Stackelberg model *à la* Daughety (1990), without assuming any specific form for the demand function and for the cost functions. We consider the relationship between the sequential Stackelberg equilibrium and the equilibrium of the simultaneous move game for

[1] Introducing an asymmetry in costs between firms could allow to endogenize their type. Similarly, the timing of positions is often exogenous in Stackelberg competition, while it can be grounded on costs asymmetries (Van Damme and Hurkens (1999)).

[2] By constant conjectural variations we mean that the formation of conjectures do not vary with a change in the strategy of a leader.

this same basic economy in case of asymmetric conjectural variations. We then extend a literature which considers conjectural variations as useful tools to easily capture in a general model various oligopoly configurations (perfect, Cournotian and collusive competitions). A synthesis of this approach has been provided by Dixit (1986), in which the Stackelberg equilibrium has not been studied. Two propositions are proved. We first show that there exists a continuum of conjectural variations such that the conjectural equilibrium locally coincides with the Stackelberg equilibrium (Proposition 1). Second, we define the conditions under which a conjectural equilibrium is a locally consistent equilibrium (i.e. such that conjectures are fulfilled). The concept of (local) consistency is restricted to firms making conjectures. Two conditions on consistency are featured: consistency within a cohort and consistency among cohorts. The Stackelberg equilibrium fulfills only the latter condition (Proposition 2). We provide an example illustrating these two propositions. Studying a linear economy, we notably show that the conjectural equilibrium may coincide with the multiple leader-follower Stackelberg equilibrium model developed by Daughety (1990). In addition, in case of asymmetry, the competitive equilibrium is locally consistent.

The paper is organized as follows. In section 2 the model is featured. Section 3 provides a definition and a characterization of the conjectural equilibrium. Section 4 states two propositions reagarding the Stackelberg equilibrium. In section 5, we give an example for a linear economy to illustrate both propositions. In section 6, we conclude.

2 The model

Consider an oligopoly industry with n firms which produce an homogeneous good. There are two types of firms, labeled 1 and 2, so the industry includes n_1 firms of type 1 and n_2 firms of type 2, with $n_1 + n_2 = n$. Type 1 firms make conjectures regarding the reaction of other type 1 firms as well as type 2 firms to a change in their strategy, while type 2 firms are assumed not not to make any conjecture (see thereafter).

Let $p(X)$ be the price function for the industry (the inverse of the market demand function), where X denotes industry output. We assume $\frac{dp(X)}{dX} < 0$ and $\frac{d^2p(X)}{dX^2} \leq 0$. Let x_1^i and x_2^j represent respectively the amounts of good produced by firm i, $i = 1, ..., n_1$, and j, $j = 1, ..., n_2$. Assume $X = X_1 + X_2$, where $X_1 = \sum_{i=1}^{n_1} x_1^i$ and $X_2 = \sum_{j=1}^{n_2} x_2^j$, denote respectively aggregate output of type 1 firms and type 2 firms. In what follows, we also denote $X_1^{-i} = \sum_{-i} x_1^{-i}$ (resp. $X_2^{-j} = \sum_{-j} x_2^{-j}$) the output of all type 1 (resp. 2) firms but i (resp. j). In addition, it is assumed that $p(X)$ is continuous.

The cost function of any firm i or j is denoted by $c_1^i(x_1^i)$, $i = 1, ..., n_1$ or $c_2^j(x_2^j)$, $j = 1, ..., n_2$. We assume $c_1^i(.) = c_1(.)$for all i and $c_2^j(.) = c_2(.)$ for all j. In addition, $\frac{dc_1^i(x_1^i)}{dx_1^i} > 0$ ($\frac{dc_2^j(x_2^j)}{dx_2^j} > 0$) and $\frac{d^2c_1^i(x_1^i)}{d(x_1^i)^2} \geq 0$ ($\frac{d^2c_2^j(x_2^j)}{d(x_2^j)^2} \geq 0$). The profits functions of firms i and j may be written:

$$\begin{array}{l} \pi_1^i = p(X)x_1^i - c_1(x_1^i),\ \forall i = 1, ..., n_1, \\ \pi_2^j = p(X)x_2^j - c_2(x_2^j),\ \forall j = 1, ..., n_2. \end{array} \tag{1}$$

Conjectural variations formed by firm i are denoted by ν_1^i (respectively ν_2^i) for $i = 1, ..., n_1$, and characterize the beliefs of firm i as for the reaction of type 1 firms (respectively type 2 firms), to a one unit increase of its output (Bresnahan (1981), Perry (1982)). Following Perry (1982), we only consider constant conjectures, so beliefs of a firm are independent of the supply of the others. Other specifications are conceivable (Figuières et al. (2004)).

Formally, the conjectural variations may be defined as follows:

$$\begin{aligned} \nu_1^i &= \frac{dX_1^{-i}}{dx_1^i} \\ \nu_2^i &= \frac{dX_2}{dx_1^i},\ i = 1, ..., n_1, \end{aligned} \tag{2}$$

where $\nu_1^i \in [-1, n_1 - 1]$ and $\nu_2^i \in [-1, n_2]$ represent intracohort and intercohort conjectures respectively, that is conjectures formed by firm i regarding the slope of the aggregate reaction functions of type 1 firms and type 2 firms.[3] In the following, we will only focus on the symmetric case, i.e. $\nu_1^i = \nu_1$ and $\nu_2^i = \nu_2$ for $i = 1, ..., n_1$.

We must consider $\nu_1 + \nu_2 \geq -1$. Among the market outcomes usually featured, three cases are of particular interest. Perfect competition corresponds to $\nu_1 + \nu_2 = -1$. In this case, each firm of type 1 expects their direct rivals to absorb exactly its supply expansion by a corresponding supply reduction, so as to leave the price unchanged. When $\nu_1 = 0$ and $\nu_2 = 0$, firms expect no reaction to its change in supply: this is the Cournotian case. Finally, when $\nu_1 = n_1 - 1$ and $\nu_2 = n_2$, the market is collusive. Among the possible configurations, it is worth considering the cases for which type 1 firms form correct expectations.

In this model, firms - whether of type 1 or 2 - play simultaneously. The players are the firms, the strategies are their supply decisions and the payoffs are their profits.

Definition 1 *An economy is a game* Γ *corresponding to a vector of* $2n$ *components, including* n *strategies and* n *payoffs* $\Gamma \equiv \left(x_1^i, x_2^j, \pi_1^i, \pi_2^j\right)_{i=1,...,n_1}^{j=1,...,n_2}$.

[3] This property holds in case on symmetry across firms (see Dixit (1986)).

3 Conjectural equilibrium: definition and characterization

Before dealing with the characterization of the conjectural equilibrium (CE), we provide a definition of it.

3.1 Definition

Definition 2 *A conjectural equilibrium for the economy* Γ *is given by a vector of strategies* $\left(\tilde{x}_1^i, \tilde{x}_2^j\right)_{i=1,...,n_1}^{j=1,...,n_2}$ *and a 2-tuple of conjectural variations* $\boldsymbol{\nu} = (\nu_1, \nu_2)$ *such that for any* $i = 1, ..., n_1$ *and any* $j = 1, ..., n_2$*:*
(i) $\pi_1^i(\tilde{x}_1^i(\boldsymbol{\nu}), \tilde{x}_1^{-i}(\boldsymbol{\nu}), \tilde{x}_2^j(\boldsymbol{\nu})) \geq \pi_1^i(x_1^i(\boldsymbol{\nu}), \tilde{x}_1^{-i}(\boldsymbol{\nu}), \tilde{x}_2^j(\boldsymbol{\nu}))$ *for all* $x_1^i \neq \tilde{x}_1^i$
(ii) $\pi_2^j(\tilde{x}_1^i(\boldsymbol{\nu}), \tilde{x}_2^j(\boldsymbol{\nu}), \tilde{x}_2^{-j}(\boldsymbol{\nu})) \geq \pi_2^j(\tilde{x}_1^i(\boldsymbol{\nu}), x_2^j(\boldsymbol{\nu}), \tilde{x}_2^{-j}(\boldsymbol{\nu}))$ *for all* $x_2^j \neq \tilde{x}_2^j$*.*

A conjectural equilibrium is a non cooperative equilibrium of game Γ. In the equilibrium, each firm i determines a strategy $\tilde{x}_1^i$ according to the conjectures ν_1 and ν_2, such that no deviation is able to increase its profit π_1^i when the strategies of the others remain unchanged. In addition, each firm j, $j = 1, ..., n_2$ determines its optimal strategy $\tilde{x}_2^j$ by taking as given the strategies of all other firms. Firms of type 2 do not make conjectures regarding unilateral deviations. This corresponds to a Nash equilibrium conditional on expectations formation. We now provide a full characterization of a CE.

3.2 Characterization of the CE

Given (1), programs of firms i and j may be written:

$$\begin{aligned} x_1^i &\in \arg\max \; p(X)x_1^i - c_1(x_1^i),\; i = 1, ..., n_1, \\ x_2^j &\in \arg\max \; p(X)x_2^j - c_2(x_2^j),\; j = 1, ..., n_2. \end{aligned} \tag{3}$$

The first-order conditions are given by:

$$p(X) + (1 + \nu_1 + \delta)\frac{dp(X)}{dX}x_1^i - \frac{dc_1(x_1^i)}{dx_1^i} = 0, \tag{4}$$

$$p(X) + \frac{dp(X)}{dX}x_2^j - \frac{dc_2(x_2^j)}{dx_2^j} = 0, \tag{5}$$

where $\delta \equiv \frac{\partial X_2}{\partial x_1^i}$ and $\nu_1 = \frac{dX_1^{-i}}{dx_1^i} = \frac{\partial X_1^{-i}}{\partial x_1^i}$. It differs from $\frac{dX_2}{dx_1^i}$ because $\frac{dX_2}{dx_1^i} = \frac{\partial X_2}{\partial x_1^i} + \frac{\partial X_2}{\partial X_1^{-i}}\frac{dX_1^{-i}}{dx_1^i} = \delta(1 + \nu_1)$, with $\frac{\partial X_2}{\partial X_1^{-i}} = \frac{\partial X_2}{\partial x_1^i}$ from the symmetry of x_1^i and X_1^{-i} in the reaction function of type 2 firms (equation 5). By rearranging the equality, it comes that $\delta = \frac{\nu_2}{1+\nu_1}$. Finally, as $\nu_1 + \nu_2 \geq -1$, $\delta \in [-1, 0]$.

These optimal conditions above implicitly define the best response function of both types of firms: $x_1^i = \phi_1(X_1^{-i}, X_2, \boldsymbol{\nu})$ and $x_2^j = \phi_2(X_1, X_2^{-j})$. In the symmetric equilibrium, $x_1^i = x_1$ for $i = 1, ..., n_1$ and $x_2^j = x_2$ for $j = 1, ..., n_2$, so one gets $x_1 = \bar{\phi}_1(n_1, X_2, \boldsymbol{\nu})$ and $x_2 = \bar{\phi}_2(n_2, X_1)$. The expected value of δ can be deduced from ν_1 and ν_2 by differentiating $X_2 = n_2\bar{\phi}_2(n_2, X_1)$. By substituting X_2 in $\bar{\phi}_1$, we therefore obtain the equilibrium strategy of a type 1 firm: $\tilde{x}_1(\nu) = \tilde{\phi}_1(n_1, n_2, \boldsymbol{\nu})$. The equilibrium strategy of a type 2 firm is then deduced: $\tilde{x}_2 = \tilde{\phi}_2(n_1, n_2, \boldsymbol{\nu})$. And the equilibrium profits can be derived: $\tilde{\pi}_1^i = \tilde{\Pi}_1(n_1, n_2, \boldsymbol{\nu})$ for any $i = 1, ..., n_1$ and $\tilde{\pi}_2^j = \tilde{\Pi}_2(n_1, n_2, \boldsymbol{\nu})$ for any $j = 1, ..., n_2$.

4 Conjectural and Stackelberg equilibria

We now prove under which conditions the CE (locally) coincides with the Stackelberg equilibrium (SE). We first define a SE. Consider that type 1 firms denote leaders, while type 2 are the followers. So, as in Daughety (1990), the industry now includes n_1 leaders and n_2 followers, with $n_1 + n_2 = n$.

Definition 3 *A Stackelberg equilibrium is given by a $(n_1 + n_2)$-tuple of strategies $\left(\hat{x}_1^i, \hat{x}_2^j\right)_{i=1,...,n_1}^{j=1,...,n_2}$ such that for any $i = 1, ..., n_1$ and for any $j = 1, ..., n_2$*

(i) $\pi_1^i(\hat{x}_1^i, \hat{x}_1^{-i}, \hat{x}_2^j(\hat{x}_1^i, \hat{x}_1^{-i})) \geq \pi_1^i(x_1^i, \hat{x}_1^{-i}, x_2^j(x_1^i, \hat{x}_1^{-i}))$, for all $x_2^j(x_1^i, \hat{x}_1^{-i})$ and $x_1^i \neq \hat{x}_1^i$,

(ii) $\pi_2^j(\hat{x}_2^j, \hat{x}_2^{-j}) \geq \pi_2^j(x_2^j, \hat{x}_2^{-j})$ for all $x_2^j \neq \hat{x}_2^j$.

In a Stackelberg equilibrium, all firms optimize their profit function, and beliefs are self-fulfilled. Leaders and followers play a Cournot game within their respective cohort. The game is played under complete but imperfect information among leaders and among followers. However, leaders have perfect information about the reaction function of the followers. Existence and uniqueness of the SE are skipped (see DeMiguel and Xu (2009), Sherali (1984), Sherali et al. (1983)).

Proposition 1 *There exists a continuum of conjectural variations such that the conjectural equilibrium locally coincides with the Stackelberg equilibrium.*

Proof. Equations (4)-(5) implicitly determine the equilibrium strategies when firm i, $i = 1, ..., n_1$, conjectures the slopes of the aggregate reaction functions.

In a SE, the reaction function of firm j is obtained from profit maximization, for given strategies of the leader X_1, and defined by $x_2^j = \psi_2(X_1, X_2^{-j})$. In the symmetric case, the reaction function becomes: $x_2 = \bar{\psi}_2(n_2, X_1)$. The program of leader i is:

$$x_1^i \in \arg\max p(x_1^i + X_1^{-i} + \sum\nolimits_{j=1}^{n_2} \bar{\psi}_2(n_2, X_1))x_1^i - c_1(x_1^i),\ i = 1, ..., n_1.$$

For any $i = 1, ..., n_1$ and $j = 1, ..., n_2$, the two first-order conditions sequentially obtained are given by:

$$p(X) + \left(1 + \eta^i\right) \frac{dp(X)}{dX} x_1^i - \frac{dc_1(x_1^i)}{dx_1^i} = 0, \quad (6)$$

$$p(X) + \frac{dp(X)}{dX} x_2^j - \frac{dc_2(x_2^j)}{dx_2^j} = 0. \quad (7)$$

where $\eta^i = n_2 \frac{\partial \bar{\psi}_2}{\partial x_1^i}(n_2, X_1)$ is the actual slope of type 2 firms' aggregate reaction function. In equilibrium, one gets $\eta^i = n_2 \frac{\partial \bar{\psi}_2}{\partial x_1^i}(n_2, \hat{X}_1)$ for $i = 1, ..., n_1$.

Equation (7) above is identitical to equation (5). Let $\nu_1 + \delta = \eta$, that is $\nu_1 + \frac{\nu_2}{1+\nu_1} = \eta$, be the conjecture of type 1 firms in a CE. The preceding system collapses to the system defined by equations (4)-(5). Thus, the CE locally coincides with the SE. QED. ■

Remark 1 *Notice that while the two equilibria coincide for $\nu_1 = 0$ and $\nu_2 = \delta = \eta$, they also coincide for an infinite number of values for ν_1 and ν_2 such that $\nu_1 + \delta = \eta$ or $\nu_2 = \eta + (\eta - 1)\nu_1$.*

Proposition 1 states that the equilibrium outcome of a simultaneous move game collapses to the equilibrium outcome a Stackelberg sequential game in which a cohort of agents make expectations regarding the impact of their decisions on the choices of another cohort of agents. We then complete the analysis of Dixit (1986) by showing that conjectural variations can also be useful to represent the Stackelberg market outcome in a static game. When embodying intercohort conjectural variations, the framework exhibits a large number of solutions corresponding to the Stackelberg market outcome.

We now focus on locally consistent CE.[4]

Definition 4 *A locally intercohort-consistent conjectural equilibrium for Γ is a CE with $\nu_2 = n_2 \frac{d\bar{\phi}_2}{dx_1^i}(n_2, \tilde{X}_1)$ and $\delta = \frac{\nu_2}{1+\nu_1}$, where $n_2\bar{\phi}_2$ is the aggregate reaction function of type 2 firms.*

A locally intercohort-consistent CE restricts the consistency to intercohort conjectures, i.e. to type 1 firms' conjectures regarding type 2 firms' reactions. But it presumes that ν_2 must be correctly expected, without implying fulfilled conjectures on δ. It defines a partially consistent equilibrium.

Definition 5 *A locally intracohort-consistent conjectural equilibrium for Γ is a CE with $\nu_1 = \frac{d\bar{\Phi}_1^{-i}}{dx_1^i}(n_1, \tilde{x}_1^i, \tilde{X}_2)$, where $\bar{\Phi}_1^{-i}$ is the aggregate reaction function of type 1 firms but i.*

A locally intracohort-consistent CE restricts the consistency to intracohort conjectures; i.e. to any type 1 firm's conjectures regarding other type 1 firms' reactions. It also defines a partially consistent equilibrium.

Definition 6 *A locally consistent conjectural equilibrium for Γ is an intracohort and intercohort consistent CE.*

A locally consistent CE is an equilibrium strategy $\tilde{x}^i$ for each i such that no firm perceives an incentive to change its supply, which is based on conjectural variations, assumed to be a correct assessment of type 1 and type 2 firms. Each firm i is then able to correctly perceive the equilibrium values of the slopes of the two aggregate reaction functions.

[4] In an oligopoly framework, local consistency has been defined by Bresnahan (1981), Perry (1982) and Figuières et al. (2004).

Proposition 2 *Within the set of CE which coincide with the SE, a necessary condition for a SE to be a locally consistent CE is $\nu_1 = 0$. In that case, $\nu_2 = \eta = n_2 \frac{\partial \bar{\psi}_2}{\partial x_1^i}(n_2, \hat{X}_1) = -1$. Then the SE converges toward the competitive equilibrium.*

Proof. According to def. 4 and 5, a locally consistent CE satisfies $\nu_1 = \frac{\partial \bar{\Phi}_1^{-i}}{\partial x_1^i}(n_1, \tilde{x}_1^i, \tilde{X}_2)$ and $\nu_2 = n_2 \frac{\partial \bar{\phi}_2}{\partial x_1^i}(n_2, \tilde{X}_1)$. Equations (5) and (7) being identical, the aggregate reaction function of type 2 firms must be equivalent in both the CE and the SE. As a consequence, in the equilibrium, their slopes must be equal; i.e. $\nu_2 = n_2 \frac{\partial \bar{\phi}_2}{\partial x_1^i}(n_2, \tilde{X}_1) = n_2 \frac{\partial \bar{\psi}_2}{\partial x_1^i}(n_2, \tilde{X}_1) = \eta$.

To construct the locally consistent CE, we must determine the best responses of all type 1 firms but i and of all type 2 firms. We follow a procedure given by Perry (1982) for the oligopoly case. The aggregate reaction functions of type 1 firms but i and type 2 firms are implicitly obtained from the first-order conditions (4) and (5) of firms $-i$ and j respectively:

$$p(X) + (1 + \nu_1 + \delta) \frac{dp(X)}{dX} \frac{X_1^{-i}}{n_1 - 1} - \frac{dc_1\left(\frac{X_1^{-i}}{n_1-1}\right)}{dx_1^{-i}} = 0,$$

$$p(X) + \frac{dp(X)}{dX} \frac{X_2}{n_2} - \frac{dc_2\left(\frac{X_2}{n_2}\right)}{dx_2} = 0,$$

where $X_2 = n_2 \bar{\phi}_2(n_2, X_1)$, and $X_1^{-i} \equiv \bar{\Phi}_1^{-i}(n_1, x_1^i, X_2)$.

Differentiating implicitly the preceding equations, one gets in the symmetric equilibrium:

$$\begin{aligned}
\frac{dX_1^{-i}}{dx_1^i} &= -\frac{(1+\delta)\left[\frac{dp(X)}{dX} + (1+\nu_1+\delta)\frac{d^2p(X)}{dX^2}\frac{\tilde{X}_1^{-i}}{n_1-1}\right]}{\frac{n_1(1+\delta)+\nu_1}{n_1-1}\frac{dp(X)}{dX} + (1+\delta)(1+\delta+\nu_1)\frac{d^2p(X)}{dX^2}\frac{\tilde{X}_1^{-i}}{n_1-1} - \frac{1}{n_1-1}\frac{d^2c_1}{d(x_1^i)^2}}, \\
\frac{dX_2}{dx_1^i} &= -\frac{\left[\frac{dp(X)}{dX} + \frac{d^2p(X)}{dX^2}\left(\frac{\tilde{X}_2}{n_2}\right)\right]\left(1 + \frac{dX_1^{-i}}{dx_1^i}\right)}{\left(\frac{n_2+1}{n_2}\right)\frac{dp(X)}{dX} + \frac{d^2p(X)}{dX^2}\left(\frac{\tilde{X}_2}{n2}\right) - \frac{1}{n_2}\frac{d^2c_2}{d(x_2^j)^2}}.
\end{aligned}$$

According to Definitions 4 and 5, a consistent CE must satisfy the next three conditions:

$$\begin{aligned}
&\text{(C1)} \quad \frac{dX_1^{-i}}{dx_1^i} = \nu_1 \\
&\text{(C2)} \quad \frac{dX_2}{dx_1^i} = \nu_2 = \eta \\
&\text{(C3)} \quad \nu_2 = \delta(1+\nu_1).
\end{aligned}$$

According to Proposition 1, the conjectural equilibria which coincide with the SE, are defined by $\nu_1 + \delta = \eta$. This equality and condition (C2) are jointly satisfied provided $\nu_1 = 0$ or $\nu_1 = -1+\eta < -1$. As $\nu_1 \in [-1, n_1-1]$, a consistent SE must satisfy:

$$\begin{aligned} \nu_1 &= 0 \\ \nu_2 &= \delta = \eta. \end{aligned}$$

From Def. 4 and 5, consistent conjectural equilibria are fixed points of:

$$\begin{aligned} \frac{dX_1^{-i}}{dx_1^i} &= \nu_1 \\ \frac{dX_2}{dx_1^i} &= \nu_2. \end{aligned}$$

Since $\frac{d^2P(X)}{dX^2} \leq 0$, the condition $\frac{dX_1^{-i}}{dx_1^i} = \nu_1 = 0$ requires $\delta = -1$. This corresponds to a specific case of perfect competition in which type 1 firms do not react to any move of another type 1 firm, while type 2 firms determine their strategy without modifying the equilibrium price. QED. ■

Proposition 2 puts into light the consistency of the CE and its correspondence with the SE: consistency of the SE requires that the firms to correctly perceive the true slope of the aggregate reaction function emanating from the firms which do not form conjectures. For $\delta \neq -1$, none of the intercohort-consistent CE that coincide with the SE can be intracohort-consistent.

5 An example

It is assumed that $p(X)$ is continuous, linear and decreasing with X and that it may be written:

$$p(X) = \max\{0, a - bX\}, \; a, b > 0. \tag{8}$$

The cost functions of any firm i and j are assumed to be linear, i.e. $c_1 x_1^i$, $\forall i = 1, ..., n_1$ and $c_2 x_2^j$, $\forall j = 1, ..., n_2$. We assume $c_1 = c_2 = c$, $\forall i, \forall j$. The profits of any firms i and any firm j may be written $\pi_1^i = \left(a - c - b(x_1^i + X_1^{-i} + X_2)\right) x_1^i$ for $i = 1, ..., n_1$ and $\pi_2^j = (a - c - b(X_1 + x_2^j + X_2^{-j}))x_2^j$ for $j = 1, ..., n_2$. This economy has a unique symmetric competitive equilibrium given by $\bar{x}_1^i = \bar{x}_2^j = \frac{a-c}{bn}$, $i = 1, ..., n_1$ and $j = 1, ..., n_2$, with $\bar{\pi}_1^i = 0$ for $i = 1, ..., n_1$ and $\bar{\pi}_2^j = 0$ for $j = 1, ..., n_2$.

Assume symmetry among all firms of each type, i.e. $x_1^i = x_1$ for any $i = 1, ..., n_1$ and $x_2^j = x_2$ for any $j = 1, ..., n_2$. The best response function of all type 1 firms but i may be deduced from (4) and the best response of any type 2 firm is given by (5):[5]

[5] One cannot use the first-order condition to firm i's profit maximization problem to define how all firms but i respond to a one-unit increase in i's output, because doing so would ignore the indirect effects of a one-unit increase in i's output.On this point, see notably Kamien and Schwartz (1983) and Perry (1982)).

$$(a - bX) - b(1 + \nu_1 + \delta)\left(\frac{X_1^{-i}}{n_1 - 1}\right) - c = 0, \tag{9}$$

$$x_2 = \frac{a - c}{b(n_2 + 1)} - \frac{X_1}{n_2 + 1}. \tag{10}$$

From (10), one deduces the slope of the reaction function of type 2 firms:[6]

$$\frac{dX_2}{dx_1^i} = \delta\left(1 + \frac{dX_1^{-i}}{dx_1^i}\right), \text{ with } \delta = -\frac{n_2}{n_2 + 1}. \tag{11}$$

In addition, differentiating (9) leads to:

$$\frac{dX_1^{-i}}{dx_1^i} = -\frac{(n_1 - 1)\left(1 + \frac{dX_2}{dx_1^i}\right)}{n_1 + \nu_1 + \delta}. \tag{12}$$

The intracohort-consistency condition $\frac{dX_1^{-i}}{dx_1^i} = \nu_1$ yields:

$$1 + \nu_2 + \frac{n_1 + \nu_1 + \delta}{n_1 - 1}\nu_1 = 0. \tag{13}$$

From the second consistency condition $\nu_2 = \delta(1 + \nu_1)$, one deduces after rearrangement:

$$\nu_1^2 + (1 + \delta)n_1\nu_1 + (1 + \delta)(n_1 - 1) = 0. \tag{14}$$

From equation (14), the locally consistent conjectural equilibrium is unique and defined by:[7]

$$\begin{aligned} \delta &= -\frac{n_2}{n_2 + 1}, \\ \nu_1 &= -\frac{(1 + \delta)n_1}{2} + \frac{\sqrt{[(1 + \delta)n_1]^2 - 4(n_1 - 1)(1 + \delta)}}{2}, \\ \nu_2 &= \delta(1 + \nu_1). \end{aligned} \tag{15}$$

It has been shown in proposition 2 that within the set of CE equilibria that coincide with the SE, a consistent equilibrium must satisfy $\nu_1 = 0$. In the above equations, there are only two cases for which $\nu_1 = 0$: either $n_1 = 1$ (the leader is alone and behaves as a monopolist) or $\delta = -1$ (perfectly competitive behaviors are expected for type 2 agents, which corresponds to $n_2 \to$

[6] When $n_2 = 1$, one has $\nu_2 = -\frac{1}{2}$, which is the true slope of the reaction function in case of a linear Stackelberg duopoly.

[7] As $\nu_1 + \nu_2 \geq -1$ one gets $\nu_1 \geq -(1 + \delta)$. Thus for $n_1 \geq 2$, one has $-\frac{(1+\delta)n_1}{2} - \frac{\sqrt{[(1+\delta)n_1]^2 - 4(n_1-1)(1+\delta)}}{2} < \frac{(1+\delta)n_1}{2} \leq -(1 + \delta)$. One deduces that the only possible value is $\nu_1 = -\frac{(1+\delta)n_1}{2} + \frac{\sqrt{[(1+\delta)n_1]^2 - 4(n_1-1)(1+\delta)}}{2}$.

∞). The SE is only partially consistent, according to definitions 4 and 5. The SE is a locally intercohort-consistent CE if $\nu_2 = -\frac{n_2}{n_2+1}$ for any feasible values of ν_1 and δ, while it is a locally intracohort-consistent CE when $\nu_1 = -\frac{(1+\delta)n_1}{2} + \frac{\sqrt{[(1+\delta)n_1]^2 - 4(n_1-1)(1+\delta)}}{2}$, for any feasible values of ν_2 and δ. It is worth noting that proposition 1 is also satisfied for $\nu_1 = 0$ and $\nu_2 = -\frac{n_2}{n_2+1}$.

Finally, from (9)-(10), the equilibrium strategies are:

$$\tilde{x}_1(\nu_1, \nu_2) = \frac{a-c}{b[n_1 + (n_2+1)(1+\nu_1+\delta)]}, \tag{16}$$

$$\tilde{x}_2(\nu_1, \nu_2) = \frac{(a-c)(1+\nu_1+\delta)}{b[n_1 + (n_2+1)(1+\nu_1+\delta)]}. \tag{17}$$

When $\nu_1 = 0$ and $\delta = -\frac{n_2}{n_2+1}$, these equilibrium strategies become:

$$\tilde{x}_1 = \frac{a-c}{b(n_1+1)}, \tag{18}$$

$$\tilde{x}_2 = \frac{(a-c)}{b(n_1+1)(n_2+1)}. \tag{19}$$

These equilibrium strategies coincide with the strategies obtained in the multiple leader-follower Stackelberg equilibrium developed by Daughety (1990).

In addition, when $\nu_1 = -1$ and $\nu_2 = \delta = 0$, (11)-(12) yields:

$$\begin{aligned} \frac{dX_1^{-i}}{dx_1^i} &= -1, \\ \frac{dX_2}{dx_1^i} &= 0. \end{aligned} \tag{20}$$

So, the asymmetric competitive equilibrium is locally consistent. Note that it does not imply that δ is correctly expected (since $\eta = \frac{-n_2}{1+n_2}$). It corresponds to competitive equilibrium strategies, where type 1 firms share the market. This result confirms that perfect competitive equilibrium is consistent when agents form competitive conjectural variations under constant marginal costs (Bresnahan (1981), Perry (1982)).

6 Conclusion

We determine the conditions under which the Stackelberg equilibrium coincides with the equilibrium of a simultaneous move game in which firms form asymmetric conjectures. We also precise the definitions of consistent conjectural variations in an asymmetric framework.

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