

Volume 31, Issue 1

The necessary condition for stability in Tobin's Walras-Keynes-Phillips model: A note

Subrata Guha

CESP, Jawaharlal Nehru University, New Delhi, India

Abstract

In a 1975 paper entitled 'Keynesian Models of Recession and Depression' James Tobin sought to formalize in a dynamic model Keynes' argument that unemployment could persist in an economy with flexible wages and prices. In the course of his analysis Tobin presented, without proof, a 'critical necessary condition for stability' of the full employment equilibrium which is violated when expenditure is sufficiently responsive to the expected inflation rate. This note contends that later attempts in the literature to prove that Tobin's condition is necessary for stability have been unsuccessful and provides a valid proof of the same result. The note also demonstrates the existence of a sufficient condition for instability of equilibrium which is weaker than the negation of Tobin's condition.

Comments from an anonymous referee of the journal are gratefully acknowledged. Thanks to Robert Dimand for responding promptly to my request for a copy of his paper with Ryan Bruno. Thanks also to the 2010-11 batch of EP 601 students in CESP who were the first to scrutinise the main argument. I, however, remain solely liable for any residual errors.

Citation: Subrata Guha, (2011) "The necessary condition for stability in Tobin's Walras-Keynes-Phillips model: A note", *Economics Bulletin*, Vol. 31 no.1 pp. 66-74.

Submitted: Nov 05 2010. **Published:** January 04, 2011.

1. Introduction: The stability condition in Tobin's W-K-P Model

Keynesian claims about the persistence of unemployment caused by shortage of aggregate demand have typically been associated with arguments relying on downward rigidity of money wages and prices. According to the neoclassical synthesis which prevailed in macroeconomics in the 1960s, Keynesian unemployment could persist with flexible wages and prices only in the extreme cases of a liquidity trap or an interest-inelastic schedule for investment, accompanied in both cases by a weak real balance effect. Subsequent attempts to locate Keynesian unemployment within macroeconomic models with micro-foundations, which characterise the non-Walrasian or disequilibrium approach and the later New Keynesian approach, also rely crucially on stickiness of wage and prices. If inflexibility of nominal wages and prices is seen as a *necessary* condition for the persistence of Keynesian unemployment then, simply from this perspective, wage and price deflation should be welcomed in situations of high unemployment.¹ There is, however, an alternative Keynesian view on unemployment and deflation, shared by Keynes himself and defended by Tobin (1975, 1992, 1993, 1997), which argues that the problem of unemployment can be exacerbated by a deflation of nominal wages and prices.

In Chapter 19 of the *General Theory* Keynes (1936) outlined a mechanism whereby reductions in money wages and prices, arising in response to deficiency of aggregate demand and consequent unemployment, could, instead of moving the economy towards a full employment equilibrium, lead to more unemployment. Falling prices could generate expectations about further falls in the price level and adversely affect spending, possibly overriding the positive effects on expenditure of a lower nominal interest rate (the Keynes Effect) and higher real balances (the Pigou Effect). Tobin (1975) attempted to formalize this argument by demonstrating how the full employment zero inflation equilibrium in a dynamic macroeconomic model (which Tobin named the *Walras-Keynes-Phillips* (W-K-P) *Model* could be unstable, once one allowed for the effects of changes in the expected rate of inflation on aggregate expenditure in the economy. A less than expected rate of inflation, following from an aggregate demand shock, could lead to a fall in the expected rate of inflation and a rise in the real interest rate with adverse effects on aggregate demand and output (the Tobin-Mundell effect).

The W-K-P model is specified as follows:

$$\dot{Y} = A_Y (E(Y, p, x) - Y) \quad (1)$$

$$\dot{p} = p \{A_P (Y - Y^*) + x\} \quad (2)$$

$$\dot{x} = A_x \left(\frac{\dot{p}}{p} - x \right) \quad (3)$$

where A_Y, A_p, A_x are positive constants, Y is aggregate real output, p is the price level, x is the expected rate of inflation, $E(\cdot)$ gives the value of aggregate real effective demand and Y^* is the value of Y at full employment. $(Y^*, p^*, 0)$ is the equilibrium of the system where p^* is the unique value of p for which $E(Y^*, p, 0) = Y^*$. Note that the Keynes Effect and the Pigou Effect imply that $E_p < 0$ whereas a positive relationship between the expected rate of inflation and current expenditures (the Tobin-Mundell Effect) implies that $E_x > 0$. Tobin also assumed that $0 < E_Y < 1$.

¹See, for example, Hall (1975).

In discussing the stability of the full-employment equilibrium in the W-K-P model, Tobin (1975, pp.199) asserted that ‘(T)he critical necessary condition for stability is:’

$$p^* E_p^* + A_x E_x^* < 0 \quad (4)$$

where $E_i^* = E_i(Y^*, p^*, 0)$ ($i = p, x$). This suggests that with no accounting of the effects of the expected rate of inflation on aggregate demand ($E_x^* = 0$) the necessary condition for stability is satisfied whereas when such effects are accounted for and significant ($E_x^* \geq -(p^* E_p^*) / A_x$) the equilibrium is unstable and unemployment can persist in the model even in the presence of price flexibility.²

Interest in Tobin’s model has revived in the context of the recent recession and the subsequent slow recovery in the United States. Palley (2008) extends Tobin’s W-K-P model to introduce additional channels through which deflation ‘exacerbates Keynesian unemployment.’ Palley (2008, pp.170) assumes, following Tobin, that the inequality in (4) gives the necessary condition for stability, arguing that ‘(S)tability requires that the Pigou and Keynes effects dominate the Tobin-Mundell effect.’ Much of his subsequent analysis is then focussed on ‘how this condition is impacted as the structure of the model is changed.’

Condition (4) has also been used to argue that the model displays a ‘corridor of stability’ in the sense that the economy will be self-correcting in the presence of small demand shocks but large shocks could lead the economy into depression (Dimand, 2005; Bruno and Dimand, 2009). The notion of a ‘corridor’ was initially proposed by Leijonhufvud (1973) to denote the limited neighbourhood of stability of a full employment equilibrium which is locally stable but globally unstable. The fact that an equilibrium is locally stable but globally unstable cannot be established simply by considering conditions for local stability.³ However, if one interprets these ‘demand shocks’ as representing non-transitory contractions in some autonomous component of aggregate demand then the argument remains valid in a broader sense. To see how, consider an economy in full-employment equilibrium. Imagine two cases of contraction in some autonomous component of aggregate demand – one relatively large, the other relatively small. The new equilibrium price level (new value of p^*) must be lower for the larger contraction and must be associated with a lower nominal interest rate. If the Keynes Effect is weaker at lower interest rates then E_p^* (in the new equilibrium) will also be smaller in absolute value in case of the larger contraction. Then, it is possible that while the necessary condition for stability of the new equilibrium is satisfied in case of the smaller contraction, it is violated in case of the larger one. In the latter case the economy fails to converge to the full-employment equilibrium.

²The implications of (4) for the stability of macroeconomic equilibrium constituted an important part of Tobin’s defence of ‘Old Keynesian’ economics almost two decades later (Tobin, 1992; Tobin, 1993).

³Tobin (1975) discussed global stability properties of the equilibrium $(Y^*, p^*, 0)$ simply by considering the possible sign of $p^* E_p(Y, p, x) + A_x E_x(Y, p, x)$ for various (Y, p, x) outside of equilibrium. However, as he acknowledged in a footnote (pp.201), Solow had pointed out to him that such discussions could only be suggestive and global properties of the system required further investigation.

The significance of Tobin's W-K-P model and the conclusions drawn by Tobin and subsequent commentators from the model depend crucially on the veracity of Tobin's claim that (4) is a necessary condition for stability of the full-employment equilibrium in the model. In effect Tobin's claim is that the equilibrium of the W-K-P model is unstable if condition (4) is violated. It therefore provides a sufficient condition for *instability* of equilibrium. In section 2 of this note we consider the contributions of McDonald (1980) and Bruno and Dimand (2009), both of whom derive a sufficient condition for *stability* of the equilibrium in the W-K-P model, but, fail to either corroborate or reject Tobin's claim about condition (4). In section 3, therefore, we derive a necessary condition for stability of the equilibrium in the W-K-P model and prove that (4) must hold for this condition to be satisfied. This provides a proof of the validity of Tobin's claim. Moreover, the negation of the necessary condition derived in section 3 provides a sufficient condition for instability of equilibrium which is weaker than that provided by the negation of Tobin's condition (4). The final section of the note carries some concluding comments.

2. Attempts at formal derivation of the stability condition

There have been at least two attempts in the literature to derive Tobin's necessary condition for stability. Palley (2008) refers to the more recent attempt by Bruno and Dimand (2009). Bruno and Dimand's derivation follows essentially the same approach as an older attempt by McDonald (1980). In both contributions the Jacobian for the system (1)–(3) is first evaluated at the equilibrium point:

$$J(Y^*, p^*, 0) = \begin{pmatrix} A_Y(E_Y^* - 1) & A_Y E_p^* & A_Y E_x^* \\ p^* A_p & 0 & p^* \\ A_x A_p & 0 & 0 \end{pmatrix}$$

where $E_i^* = E_i(Y^*, p^*, 0)$ ($i = Y, p, x$). The equilibrium $(Y^*, p^*, 0)$ is stable if all characteristic roots of $J(Y^*, p^*, 0)$ have negative real parts and it is unstable if at least one characteristic root has a positive real part. However, we cannot conclude anything about the stability of the equilibrium from the characteristic roots of $J(Y^*, p^*, 0)$ if all the characteristic roots have non-negative real parts but there is at least one characteristic root with real part equal to zero.⁴

Both McDonald (1980) and Bruno and Dimand (2009) then proceed to derive the Routh-Hurwitz stability conditions for $J(Y^*, p^*, 0)$. These are necessary and sufficient conditions for all characteristic roots of $J(Y^*, p^*, 0)$ to have negative real parts. If

$$\alpha_0 \lambda^3 + \alpha_1 \lambda^2 + \alpha_2 \lambda + \alpha_3 = 0$$

is the characteristic equation of a three-dimensional square matrix written such that $\alpha_0 > 0$ then the Routh-Hurwitz conditions for that matrix are

$$\alpha_1 > 0, \quad \begin{vmatrix} \alpha_1 & \alpha_0 \\ \alpha_3 & \alpha_2 \end{vmatrix} > 0 \quad \text{and} \quad \begin{vmatrix} \alpha_1 & \alpha_0 & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 \\ 0 & 0 & \alpha_3 \end{vmatrix} > 0. \quad ^5$$

⁴See, for example, Braun (1993), pp.386, and Terrell (2009), pp.173-174.

In case of the matrix $J(Y^*, p^*, 0)$, $\alpha_0 = 1$, $\alpha_1 = A_Y(1 - E_Y^*)$, $\alpha_2 = -A_Y A_p (p^* E_p^* + A_x E_x^*)$ and $\alpha_3 = -A_Y A_p A_x p^* E_p^*$. Since A_Y, A_p, A_x are positive constants, $E_Y^* < 1$ and $E_p^* < 0$, the assumptions of the model imply that $\alpha_1 > 0$ and $\alpha_3 > 0$. Therefore, the Routh-Hurwitz conditions for $J(Y^*, p^*, 0)$ are satisfied and all the characteristic roots of $J(Y^*, p^*, 0)$ have negative real parts if and only if $\alpha_1 \alpha_2 - \alpha_0 \alpha_3 > 0$, i.e.

$$-A_Y(1 - E_Y^*)(E_p^* p^* + A_x E_x^*) + A_x p^* E_p^* > 0 \quad (5)$$

Since the equilibrium $(Y^*, p^*, 0)$ is stable *if* all characteristic roots of $J(Y^*, p^*, 0)$ have negative real parts, both McDonald (1980) and Bruno and Dimand (2009) in effect proved that (5) is a *sufficient* condition for stability of the equilibrium $(Y^*, p^*, 0)$. McDonald pointed out that this is a more stringent condition than Tobin's condition (4) requiring not only that $E_x^* < -(p^* E_p^*)/A_x$ but also that

$$E_x^* < \frac{-p^* E_p^* \{A_Y(1 - E_Y^*) - A_x\}}{A_Y(1 - E_Y^*) A_x} \quad (6)$$

Note, for example, that (6) is satisfied only if $A_x < A_Y(1 - E_Y^*)$. McDonald argued on this basis that 'Tobin's statement of the stability condition of the W-K-P model is incorrect ...' (McDonald, 1980; pp.829).

While McDonald refers to inequality (5) simply as a 'stability condition', the procedure used by McDonald to derive (5) proves only that (5) is a *sufficient* condition for stability. Tobin's claim, on the other hand, was that inequality (4) is a *necessary* condition for stability.⁶ In other words, while McDonald proved that the equilibrium of the W-K-P model is *stable* if (5) is satisfied, Tobin's claim was that the equilibrium is *unstable* if (4) is violated. McDonald's claim about the incorrectness of 'Tobin's stability condition' is therefore itself incorrect unless (5) can also be proved to be a necessary condition for stability. McDonald (1980) did not provide this proof.

Bruno and Dimand (2009), in their derivation, argue that Tobin's condition (4) is a necessary condition for (5) to be satisfied, 'which is why Tobin referred to [it] ... as the critical *necessary* condition.' Note however that (4) is only a necessary condition for (5) to be satisfied and is a 'necessary condition for stability' (as claimed by Tobin) only if (5) itself is a necessary condition for stability. Since the derivation of (5) by Bruno and Dimand followed the same procedure used by McDonald, they proved the sufficiency but not the necessity of (5) for stability of equilibrium in the W-K-P model.

Thus, in trying to evaluate Tobin's claim about (4) being a *necessary* condition for stability of equilibrium in the W-K-P model, both McDonald (1980) and Bruno and Dimand (2009) proved that the Routh-Hurwitz stability conditions for the equilibrium are satisfied if and only if (5) is satisfied. Satisfaction of the Routh-Hurwitz conditions implies that the characteristic roots of $J(Y^*, p^*, 0)$ all have negative real parts and the equilibrium $(Y^*, p^*, 0)$ is asymptotically stable. Violation of the Routh-Hurwitz conditions,

⁵Given that $\alpha_0 > 0$, this is equivalent to the set of conditions: $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_3 > 0$, $\alpha_1 \alpha_2 - \alpha_0 \alpha_3 > 0$. See, for example, Takayama (1994), pp.344.

⁶This was pointed out by Takayama (1994). See pp.369, footnote 43.

however, does not preclude the possibility that all characteristic roots of $J(Y^*, p^*, 0)$ have non-negative real parts and the equilibrium $(Y^*, p^*, 0)$ is either Liapunov stable or asymptotically stable. McDonald as well as Bruno and Dimand therefore proved that (5) is a *sufficient* condition for stability but failed to establish any set of conditions as being *necessary* for stability. Consequently, they failed to either prove or disprove Tobin's claim that violation of condition (4) implies that the macroeconomic equilibrium in the W-K-P model is unstable.

3. The necessity of Tobin's condition for stability of equilibrium

In this section we prove that Tobin's condition (4) is a *necessary* condition for stability of equilibrium in the W-K-P model. The proof of this proposition (see below) is based on the fact that if we replace the strict inequality in (5) with a weak inequality we get a necessary condition for stability of equilibrium in the system (1)–(3).

Proposition 1 *A necessary condition for local asymptotic stability of the equilibrium $(Y^*, p^*, 0)$ of the system (1)–(3) is that*

$$E_p^* p^* + A_x E_x^* < 0$$

Proof. Suppose the equilibrium $(Y^*, p^*, 0)$ of the system (1)–(3) is hyperbolic (that is, all characteristic roots of $J(Y^*, p^*, 0)$ have non-zero real parts).

In this case there are two possibilities: All characteristic roots of $J(Y^*, p^*, 0)$ have negative real parts and the equilibrium is locally asymptotically stable or at least one characteristic root has a positive real part and the equilibrium is unstable (Hirsch and Smale, 1974)⁷. From McDonald (1980) and Bruno and Dimand (2009) we know that (5) is a necessary and sufficient condition for all characteristic roots of $J(Y^*, p^*, 0)$ to have negative real parts. Therefore:

If the equilibrium $(Y^*, p^*, 0)$ is hyperbolic it is locally asymptotically stable
if and only if condition (5) is satisfied (7)

Next we derive the exact condition under which the equilibrium $(Y^*, p^*, 0)$ of the system (1)–(3) is not hyperbolic (that is, 0 is a characteristic root of $J(Y^*, p^*, 0)$ or $J(Y^*, p^*, 0)$ has two purely imaginary characteristic roots).

Note that the characteristic equation for $J(Y^*, p^*, 0)$ is given by

$$\lambda^3 + A_Y (1 - E_Y) \lambda^2 - A_Y A_p (p^* E_p^* + A_x E_x^*) \lambda - A_Y A_p A_x p^* E_p^* = 0 \quad (8)$$

Since the constant term $-A_Y A_p A_x p^* E_p^*$ on the left hand side of (8) is positive, 0 cannot be a characteristic root of $J(Y^*, p^*, 0)$.

Note also that

$$\begin{aligned} \text{trace } J(Y^*, p^*, 0) &= -A_Y (1 - E_Y^*) < 0 \\ \det J(Y^*, p^*, 0) &= A_Y A_p A_x p^* E_p^* < 0 \end{aligned}$$

Since the determinant of a square matrix is equal to the product of its characteristic roots and $\det J(Y^*, p^*, 0) < 0$ it follows that at least one of the following three possibilities

⁷See Theorem and Corollary on pp.187.

must be true: (i) all three characteristic roots are real and negative, (ii) there are three real characteristic roots of which two are positive and one is negative, and (iii) there are two complex conjugate characteristic roots and one real characteristic root which is negative.

Since $\text{trace } J(Y^*, p^*, 0)$ is equal to the sum of the characteristic roots of $J(Y^*, p^*, 0)$ and is negative and at least one of the three possibilities (i), (ii) and (iii) must be true, $J(Y^*, p^*, 0)$ will have two purely imaginary characteristic roots if and only if $-A_Y(1 - E_Y^*)$ is a characteristic root of $J(Y^*, p^*, 0)$.

Since 0 cannot be a characteristic root of $J(Y^*, p^*, 0)$ it follows that the equilibrium $(Y^*, p^*, 0)$ is not hyperbolic if and only if $-A_Y(1 - E_Y^*)$ is a root of the equation (8) involving the variable λ .

It can then be shown that:

The equilibrium $(Y^*, p^*, 0)$ is not hyperbolic if and only if

$$-A_Y(1 - E_Y^*)(E_p^*p^* + A_xE_x^*) + A_xp^*E_p^* = 0 \quad (9)$$

From (5), (7) and (9) it follows that:

The equilibrium $(Y^*, p^*, 0)$ is locally asymptotically stable only if

$$-A_Y(1 - E_Y^*)(E_p^*p^* + A_xE_x^*) + A_xp^*E_p^* \geq 0$$

Since $A_xp^*E_p^* < 0$ it follows that the equilibrium $(Y^*, p^*, 0)$ is locally asymptotically stable only if

$$E_p^*p^* + A_xE_x^* < 0$$

This concludes the required proof. ■

Tobin's analysis of the factors affecting stability of macroeconomic equilibrium was carried out using (4), the negation of which is a sufficient condition for instability. McDonald (1980) proved that (5) is a sufficient condition for stability. There is no doubt, however, that claims regarding how different factors affect stability have to be carried out using both kinds of conditions. In this context it is interesting to note that conditions (5), (7) and (9) imply that the sufficient condition for instability is, in fact, weaker than the negation of condition (4). Thus, all currently known results regarding the stability of equilibrium in Tobin's W-K-P model can be brought together in the following proposition.

Proposition 2 (i) *The equilibrium $(Y^*, p^*, 0)$ of the system (1)-(3) is locally asymptotically stable if $-A_Y(1 - E_Y^*)(E_p^*p^* + A_xE_x^*) + A_xp^*E_p^* > 0$.*

(ii) *The equilibrium $(Y^*, p^*, 0)$ of the system (1)-(3) is unstable if $-A_Y(1 - E_Y^*)(E_p^*p^* + A_xE_x^*) + A_xp^*E_p^* < 0$.*

Proof. (i) follows from McDonald (1980) and (ii) follows from the proof of Proposition 1. ■

The proposition implies that if $A_x > A_Y(1 - E_Y^*)$ the equilibrium is always unstable. If $A_x \leq A_Y(1 - E_Y^*)$ then for a sufficiently large (small) absolute value of E_p^* and sufficiently small (large) values of A_x and E_x^* the equilibrium is stable (unstable). Thus, provided output adjusts sufficiently quickly to demand, the marginal propensity to spend out of income is sufficiently small and the adjustment of the expected rate of inflation to

the actual rate is sufficiently slow, both Tobin's claims regarding the destabilizing role of deflationary expectations and subsequent claims in the literature about the presence of a 'corridor of stability' in his W-K-P model are correct.

4. Conclusion

Tobin's Walras-Keynes-Phillips model has been an influential vehicle for promulgating the 'Old Keynesian' view that rigidities in nominal wages and prices are *not* responsible for Keynesian unemployment because wage and price flexibility may themselves have a destabilizing impact on the economy. The stability properties of the equilibrium of the model are therefore crucial for the conclusions drawn from the model.

The Routh-Hurwitz conditions applied to the Jacobian of a nonlinear system evaluated at an equilibrium point of the system provide *sufficient* conditions for local asymptotic stability of that equilibrium. Thus, proving that a condition is necessary for satisfaction of the Routh-Hurwitz conditions is not enough to prove that the condition is necessary for asymptotic stability of equilibrium. Attempts at formal derivation of Tobin's necessary condition for stability of equilibrium in his Walras-Keynes-Phillips model have floundered on this point.

The Routh-Hurwitz conditions are, however, *necessary and sufficient* for asymptotic stability of equilibrium if the equilibrium is hyperbolic in nature. In the case of Tobin's model it is possible to derive a necessary and sufficient condition for the equilibrium to be non-hyperbolic. This condition together with the necessary and sufficient condition for stability of a hyperbolic equilibrium (derived from the Routh-Hurwitz conditions) was used in this note to obtain a necessary condition for stability of equilibrium. Tobin's stability condition is necessary for satisfaction of this necessary condition and therefore necessary for stability of the equilibrium in the W-K-P model. If we assume that the speed of adjustment of output to demand is sufficiently large and the marginal propensity to spend out of income in the economy is sufficiently small, both Tobin's claims regarding the destabilizing role of deflationary expectations and subsequent claims in the literature about the presence of a 'corridor of stability' in Tobin's W-K-P model are substantiated.

References

- Braun, Martin (1993) *Differential Equations and Their Applications*, Berlin: Springer.
- Bruno, Ryan and Robert W. Dimand (2009) "The Corridor of Stability in Tobin's Keynesian Model of Recession and Depression," *International Journal of Applied Economics and Econometrics* **17**, 17-25.
- Dimand, Robert W. (2005) "Fisher, Keynes, and the Corridor of Stability," *American Journal of Economics and Sociology* **64**, 185-199.
- Hall, Robert E. (1975) "The Rigidity of Wages and the Persistence of Unemployment," *Brookings Papers on Economic Activity* **2**, 301-335.
- Hirsch, Morris W. and Stephen Smale (1974) *Differential Equations, Dynamical Systems, and Linear Algebra*, New York: Academic Press.

Keynes, John Maynard (1936) *The General Theory of Employment, Interest and Money*, London: Macmillan.

Leijonhufvud, Axel (1973) "Effective Demand Failures," *Swedish Journal of Economics* **75**, 27-48.

McDonald, Ian M. (1980) "On the Comparison of the Stability Implications of Marshallian and Walrasian Adjustment Schemes: Note," *American Economic Review* **70**, 829-833.

Palley, Thomas I. (2008) "Keynesian Models of Deflation and Depression Revisited," *Journal of Economic Behavior and Organization* **68**, 167-177.

Takayama, Akira (1994) *Analytical Methods in Economics*, Ann Arbor: University of Michigan Press.

Terrell, William J. (2009) *Stability and Stabilization: An Introduction*, Princeton: Princeton University Press.

Tobin, James (1975) "Keynesian Models of Recession and Depression," *American Economic Review* **65**, 195-202.

Tobin, James (1992) "An Old Keynesian Counterattacks," *Eastern Economic Journal* **18**, 387-400.

Tobin, James (1993) "Price Flexibility and Output Stability: An Old Keynesian View," *Journal of Economic Perspectives* **7**, 45-65.

Tobin, James (1997) "An Overview of *The General Theory*," in *A 'Second Edition' of The General Theory, Vol.2* by G. C. Harcourt and P. A. Riach, Eds., London: Routledge.