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# Evolutionary Sequential Trading

Ryosuke Ishii
*Institute of Intellectual Property*

## Abstract

This paper analyzes an Easley and O'Hara (1992) type sequential trading model in an evolutionary setting. We assume that the memory of a market maker is limited, and that traders endogenously choose whether to acquire private information with a fixed cost. We show that the ratio of the informed traders is proportional to the width of the bid ask spread, and that the price converges to the strong-form efficient level exponentially.

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# 1 Introduction

Easley and O'Hara (1992) show how trading volume affects the speed of price adjustment in a market with asymmetric information. They have the following setting. A group of informed traders is fixed and they have identical private information about the value of the asset. Other traders do not have such private information. At each time, a randomly chosen trader can place one unit of sell or buy order. A market maker determines the price without knowing whether the trader has the private information or not. The market maker learns the traders' private information from the trading history, so that the bid-ask spread (resp. the price) converges to zero (the strong-form efficient level[1]) exponentially over time.

They assume that the ratio of informed traders is fixed at some constant $\mu$ over time. However this assumption may be a simplifying assumption. Since each trader who comes to the market and trades at most once in sequence has the fixed amount of information, it might sound obvious that the price factors in the information at a rapid rate. Then we will consider the model where there are tangible and intangible costs to receive information, and where each trader decides whether to receive the information. Once we consider the cost, it is not apparent whether the price converges to the strong-form efficient level. When a chosen trader knows that the price enough reflects the information, he may not be willing to pay for the private information, which may not result in the exponential convergence shown in their paper.

Our model is not a straightforward extension of their model. To simplify the analysis, we change the setting by assuming that traders' memories are limited. This dynmic is a Markov process. We assume that the intraday trading volume is high enough, so that most of trading are done under the stationary state of this Markov process. $\mu$ is fixed throughout the day. Our model repeats these kinds of intraday trading. After the trading hour, small minority of the trader can review their choices whether to pay for the information. They are prepared for the trade in the stationary state under $\mu$ observed on the adjacent trade. New $\mu$ is determined by their choice changes, a number of trades under new $\mu$ on the next day leads to a new stationary state, and there are review of choices after trading hours... Through this process, $\mu$ should come to the level where there is no difference between pay-

[1]In the sense of their model, this term means that the price fully reflects the informed traders' information.

ing for the information and not paying. We will see the behavior of prices in the place where it settles down as the memory of the market maker becomes infinite.

We obtain the result in which the ratio of the informed traders is proportional to the width of the bid ask spread, and the price converges to the strong-form efficient level exponentially as is shown in Easley and O'Hara (1992).

# 2 Model

We consider an intraday transaction. The value of the asset at the end of the day is represented by a random variable $V$. Before the start of the trading day, an information event that relates to the asset value occurs with the probability $\alpha$, where $0 < \alpha < 1$. If the information event occurs, the fraction $\mu$ of the traders can know the occurrence of the information event, and observe an identical signal. We require that $0 < \mu < 1$. The other $1 - \mu$ traders and the market maker do not observe it. Formally, we define an information event as the occurrence of a signal $\Psi$ about $V$. The signal can take one of two values, $L$ and $H$, with probabilities $\delta > 0$ and $1 - \delta > 0$ respectively. We let the expected value of the asset conditional on the signal be $E[V|\Psi = L] = \underline{V}$ or $E[V|\Psi = H] = \overline{V}$. If no information event has occurred, we denote this as $\Psi = 0$ and the expected value of the asset simply remains at its unconditional level $V^* = \delta\underline{V} + (1-\delta)\overline{V}$.

The market maker sets the bid and ask prices. Then a randomly chosen trader comes to the market and places one unit of sell or buy order one by one.

Each informed trader is risk-neutral and take prices as given. She will buy if she has seen a high signal and an ask price below $\overline{V}$; she will sell if she has seen a low signal and a bid price above $\underline{V}$.

An trade for liquidity reasons such as the timing of consumption or portfolio considerations may arise. She will buy (resp. sell) with the constant probability $\epsilon^B > 0$ $\left(\epsilon^S > 0\right)$. If it is not the case, she does not want to trade, because the quotes should be unprofitable for her.

We assume that the market maker is risk-neutral and behaves competitively. That is, his price quotes yield zero expected profit conditional on a trade at the quotes and on the past trading information in the memory.

This trading structure can be understood more easily by the tree diagram given in Figure 1.

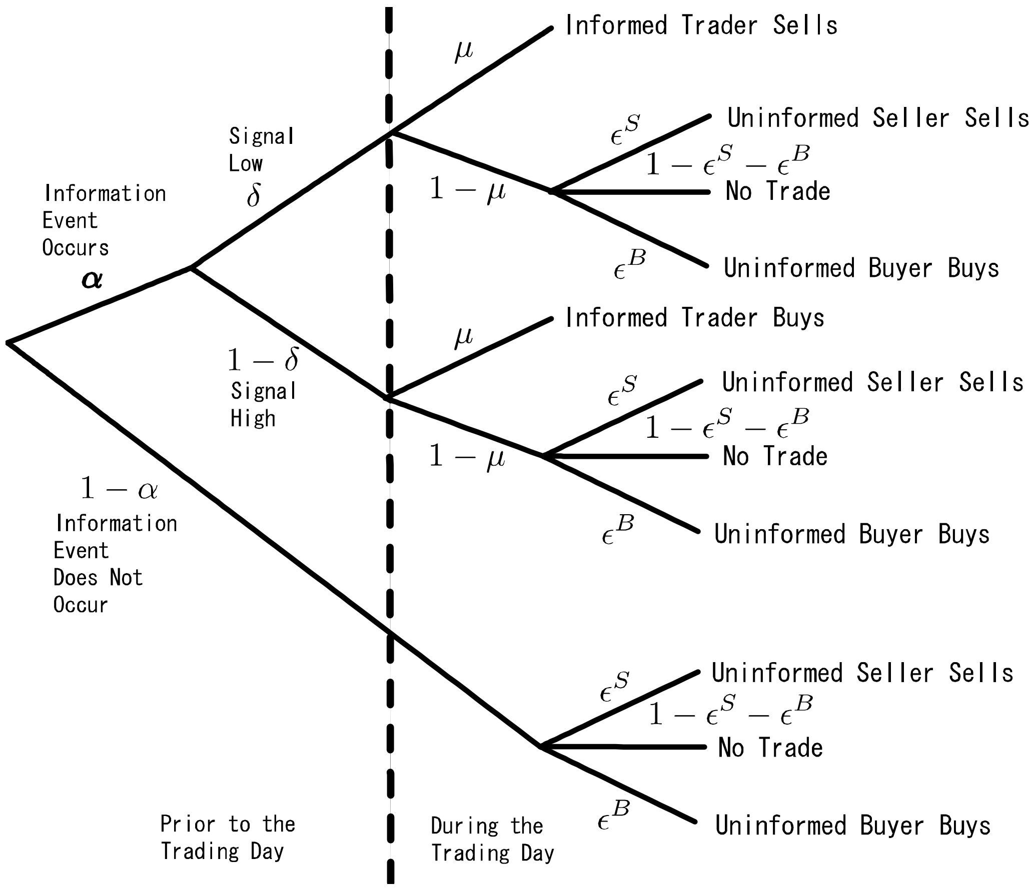


Figure 1

At the first node nature selects whether the information event occurs. If there is an information event, then the type of signal (either $L$ or $H$) is determined at the second node. These two nodes are reached once and for all at the beginning of the day. Traders are selected at each time $t$ to trade based on the probabilities described above. Thus, if an information event has occurred, an informed trader is selected with probability $\mu$, and she then chooses either to buy or sell. Similarly, with probability $1-\mu$ an uninformed trader is selected and she may choose to buy, sell or not trade with the indicated probabilities. For trade in the next time interval, only the process at the right part of dotted line on the tree diagram is repeated. This continues throughout the day.

The market maker is Bayesian who has limited memory. He does not know whether the information event has occurred, whether it is good or bad news given that it has occurred, or whether any particular trader is informed. Over time, the traders' orders allow the market maker to learn whether the information event has occurred and whether it is good or bad news given that it has occurred and revise his belief. But he cannot remember all past trades. He remembers only $m$ outcomes of the past trading. That is, he remembers the number of the buy orders $\beta$, sell orders $s$, and no trade $n$ ($\beta + s + n \leq m$) in the past trades. Note that the quotes submitted by the market maker depend not on the entire history but only on the ratios of the number of each order (See Proposition 3 in Easley and O'Hara (1992)). He can remember all the past trades before the time $m$. At $t$ ($t > m$), he randomly forgets one of $m$ outcomes in his memory at $t-1$ with equal probabilities, and remembers the new outcome at $t$.[2]

With a calculation similar to the one in Easley and O'Hara (1992), the market maker's beliefs given his memory are given by:

$$
\begin{aligned}
&\Pr\{\Psi = 0 |\, (n, s, \beta)\} \\
&= \frac{\Pr\{\Psi = 0,\ (n, s, \beta)\}}{\Pr\{\Psi = 0,\ (n, s, \beta)\} + \Pr\{\Psi = L,\ (n, s, \beta)\} + \Pr\{\Psi = H,\ (n, s, \beta)\}} \\
&= (1-\alpha)\left(\epsilon^S\right)^s \left(\epsilon^B\right)^\beta / \{(1-\alpha)\left(\epsilon^S\right)^s \left(\epsilon^B\right)^\beta + (1-\mu)^n \\
&\cdot [\alpha\delta\left(\mu + (1-\mu)\epsilon^S\right)^s \left((1-\mu)\epsilon^B\right)^\beta + \alpha(1-\delta)\left((1-\mu)\epsilon^S\right)^s \left(\mu + (1-\mu)\epsilon^B\right)^\beta]\}.
\end{aligned}
$$

The probabilities of low and high signals are calculated similarly.

Since beliefs depend on $(n, s, \beta)$, quotes will also depend on these variables. The bid $b$ and the ask $a$ can be written as:

$$
\begin{aligned}
b &= \Pr\{\Psi = L |\, (n, s+1, \beta)\}\, \underline{V} \\
&\quad + \Pr\{\Psi = H |\, (n, s+1, \beta)\}\, \overline{V} + \Pr\{\Psi = 0 |\, (n, s+1, \beta)\}\, V^*, \text{ and} \\
a &= \Pr\{\Psi = L |\, (n, s, \beta+1)\}\, \underline{V} \\
&\quad + \Pr\{\Psi = H |\, (n, s, \beta+1)\}\, \overline{V} + \Pr\{\Psi = 0 |\, (n, s, \beta+1)\}\, V^*.
\end{aligned}
$$

[2] He is not aware that he will forget one outcome. When he set the quotes, he considers the $m+1$ trade ($m$ past outcomes and a potential sell/buy/no-trade order at the present time) as is noted below.

We define the collection $\Psi$, $n, s$, and $\beta$ as a *state*. We can calculate the transition probabilities between all pairs of the states, and regard the process as Markov process. This process is not irreducible. For example, $(L, \cdot, \cdot, \cdot)$ is not accessible from $(H, \cdot, \cdot, \cdot)$, and vice versa. We can find three absorbing sets there.

$$\begin{aligned}
&\{(L, n, s, \beta)\}_{n+s+\beta=m}, \\
&\{(H, n, s, \beta)\}_{n+s+\beta=m}, \text{ and} \\
&\{(0, n, s, \beta)\}_{n+s+\beta=m}.
\end{aligned}$$

While there are infinite number of stationary distributions in this Markov process, the ratios of stationary distribution probability pairs of states are the same as the corresponding ones that have the same $\Psi$. That is, there is a degree of freedom of how to distribute gross probabilities to three absorbing states, but the allocation of the distributed probability in each absorbing set is unique. Formally, when we fix an outcome of $\Psi = L$, $H$, or $0$ arbitrary, the "partial processes," in which the sets of states consist of $n + s + \beta = m$, are finite, irreducible, and recurrent. Therefore each partial process has a unique stationary distribution. Multiplying these stationary distributions given $\Psi$ by the probabilities of $\Psi$ occurring, the distribution of the convex combination of them is also stationary, which we call *stationary state*.

# 3 Stationary State

In this section, we solve the stationary state analytically.

**Proposition 1** *The stationary state $\left(x_{(\Psi,n,s,\beta)}\right)$ is as follows:*

$$x_{(\Psi,n,s,\beta)} = \begin{cases} \alpha\delta \frac{m!}{n!s!\beta!} p_n(L)^n p_s(L)^s p_\beta(L)^\beta & \text{if} \quad \Psi = L \\ \alpha(1-\delta) \frac{m!}{n!s!\beta!} p_n(H)^n p_s(H)^s p_\beta(H)^\beta & \text{if} \quad \Psi = H, \text{ and} \\ (1-\alpha) \frac{m!}{n!s!\beta!} p_n(0)^n p_s(0)^s p_\beta(0)^\beta & \text{if} \quad \Psi = 0, \end{cases}$$

*where*

$$\begin{aligned}
p_n(\Psi) &= \Pr\{\text{No Trade}|\Psi\}, \\
p_s(\Psi) &= \Pr\{\text{Sell}|\Psi\}, \text{ and} \\
p_\beta(\Psi) &= \Pr\{\text{Buy}|\Psi\}.
\end{aligned}$$

***Proof.*** *$\left(x_{(\Psi,n,s,\beta)}\right)$ is probability distribution because $x_{(\Psi,n,s,\beta)} \geq 0$ holds for every $\Psi, n, s, \beta$, and*

$$\begin{aligned}
\sum_{\Psi,n,s,\beta} x_{(\Psi,n,s,\beta)} &= \alpha\delta \left(p_n(L) + p_s(L) + p_\beta(L)\right)^3 \\
&+ \alpha(1-\delta)\left(p_n(H) + p_s(H) + p_\beta(H)\right)^3 \\
&+ (1-\alpha)\left(p_n(0) + p_s(0) + p_\beta(0)\right)^3 \\
&= 1.
\end{aligned}$$

*The transition probabilities in both the entire process and the partial processes are*

$$\begin{aligned}
\Pr\{(\Psi,n,s,\beta) \to (\Psi,n-1,s,\beta+1)\} &= \frac{n}{m}p_\beta(\Psi), \\
\Pr\{(\Psi,n,s,\beta) \to (\Psi,n-1,s+1,\beta)\} &= \frac{n}{m}p_s(\Psi), \\
\Pr\{(\Psi,n,s,\beta) \to (\Psi,n,s-1,\beta+1)\} &= \frac{s}{m}p_\beta(\Psi), \\
\Pr\{(\Psi,n,s,\beta) \to (\Psi,n,s,\beta)\} &= \frac{n}{m}p_n(\Psi) + \frac{s}{m}p_s(\Psi) + \frac{\beta}{m}p_\beta(\Psi), \\
\Pr\{(\Psi,n,s,\beta) \to (\Psi,n,s+1,\beta-1)\} &= \frac{\beta}{m}p_s(\Psi), \\
\Pr\{(\Psi,n,s,\beta) \to (\Psi,n+1,s-1,\beta)\} &= \frac{s}{m}p_n(\Psi), \\
\Pr\{(\Psi,n,s,\beta) \to (\Psi,n+1,s,\beta-1)\} &= \frac{\beta}{m}p_n(\Psi), \text{ and} \\
\Pr\{(\Psi,n,s,\beta) \to \textit{others}\} &= 0.
\end{aligned}$$

*Therefore, for all* $n, s, \beta \in N$ $(n+s+\beta = m)$,

$$\begin{aligned}
&\sum_{(\Psi,n',s',\beta')} \Pr\left\{(\Psi,n',s',\beta') \to (\Psi,n,s,\beta)\right\} \frac{m!}{n!s!\beta!} p_n(\Psi)^n p_s(\Psi)^s p_\beta(\Psi)^\beta \\
&= \frac{m!}{(n+1)!s!(\beta-1)!} p_n(\Psi)^{n+1} p_s(\Psi)^s p_\beta(\Psi)^{\beta-1} \cdot \frac{n+1}{m} p_\beta(\Psi) \\
&+ \frac{m!}{(n+1)!(s-1)!\beta!} p_n(\Psi)^{n+1} p_s(\Psi)^{s-1} p_\beta(\Psi)^{\beta} \cdot \frac{n+1}{m} p_s(\Psi) \\
&+ \frac{m!}{n!(s+1)!(\beta-1)!} p_n(\Psi)^{n} p_s(\Psi)^{s+1} p_\beta(\Psi)^{\beta-1} \cdot \frac{s+1}{m} p_\beta(\Psi) \\
&+ \frac{m!}{n!s!\beta!} p_n(\Psi)^{n} p_s(\Psi)^{s} p_\beta(\Psi)^{\beta} \cdot \left(\frac{n}{m} p_n(\Psi) + \frac{s}{m} p_s(\Psi) + \frac{\beta}{m} p_\beta(\Psi)\right) \\
&+ \frac{m!}{n!(s-1)!(\beta+1)!} p_n(\Psi)^{n} p_s(\Psi)^{s-1} p_\beta(\Psi)^{\beta+1} \cdot \frac{\beta+1}{m} p_s(\Psi) \\
&+ \frac{m!}{(n-1)!(s+1)!\beta!} p_n(\Psi)^{n-1} p_s(\Psi)^{s+1} p_\beta(\Psi)^{\beta} \cdot \frac{s+1}{m} p_n(\Psi) \\
&+ \frac{m!}{(n-1)!s!(\beta+1)!} p_n(\Psi)^{n-1} p_s(\Psi)^{s} p_\beta(\Psi)^{\beta+1} \cdot \frac{\beta+1}{m} p_n(\Psi) \\
&= \frac{m!}{n!s!\beta!} p_n^n p_s^s p_\beta^\beta,
\end{aligned}$$

*where*

$$0! = 1, \text{ and } \frac{1}{(-1)!} = 0.$$

■

# 4 Evolutionary Decision Making on Free Entry

We now modify the setting of the model. We repeat the intraday trading analyzed above again and again. We assume that the traders can choose to receive the information by bearing cost. The ratio of traders who can receive the information is $\mu$ at the initial day. Every day one trader is randomly chosen before the trading, and she can decide whether to receive the information with the cost $c > 0$. She expects that the present state is the stationary state on that occasion. If she chooses to receive information (resp. does not receive information), the value of $\mu$ becomes greater (less) than now in a small range. If she chooses to receive the information, and if the information event occurs, then she can receive the information. Otherwise she behave as an uninformed trader. Other traders do not change their information acquisition behaviors on this day. Now we can calculate the stationary state under the new $\mu$ and the trading converges to the stationary state. On the next day, another trader is randomly chosen in the new stationary state, and she makes the same decision... We call the state where the trader is indifferent between acquiring and not acquiring the information as a *stable state*.

The following scenario may facilitate the readers' understandings. The traders often get up late. The market has already opened, and there are too many trades for the market maker to remember because of his limited memory $m$. The state in the market seems to have converged to the stationary state. The proportion $\mu$ of the traders subscribe the newspapers, and can read them before they trade. Since they get up now, they do not know if important events occurred today until they read the newspaper. The market maker is busy and has no time to read the newspaper. The fee of subscription is $c$. After reading the newspaper, she can place her order once in order to maximize her today's payoff. Even if she can not receive the information and if she does not want to trade, she may have to buy or sell with the probabilities $\epsilon^B$ or $\epsilon^S$. A newspaper gentry goes door-to-door and solicits subscriptions at a pace of a house a day (probably after the trading hours). When the newspaper gentry comes, a trader can choose whether she enters her subscription, renews, or cancels. The market maker knows how many people read the newspapers.

What is a stable state in such a dynamic? When the number of traders who read the newspapers is small (resp. large), it is difficult (easy) for the market maker to learn through the outcomes, the price reflects the information less (more), and the value of reading the newspaper is large (small), which results in the increase (decrease) of the willingness for the traders to read the newspapers. Reading and not reading must be indifferent in the stable state.

We can think of the bid-ask spread as an indicator of information efficiency. We see the relationship between the bid-ask spread and the memory length of the market maker. Now we can prove that the average value of the bid-ask spread is proportional to $\mu$ in the stable state.

**Proposition 2** *Suppose that* $\epsilon^S = \epsilon^B$*. Then, in the stable state the following holds:*

$$\sum x_{(\Psi,n,s,\beta)} (a-b) = \frac{\mu}{\epsilon^S} c,$$

*where* $a-b$ *is the bid-ask spread when the memories are* $(n,s,\beta)$*.*
***Proof.*** *Denote the expected payoff of the trader paying* $c$ *as* $X$*, and not paying as* $Y$*. Since the expected payoff of the market maker is zero,*

$$\mu X + (1-\mu) Y = 0.$$

*On the other hand, there is no difference between paying and not paying for the trader,*

$$X - c = Y.$$

*Therefore*

$$\mu c = -Y.$$

■

Since the bid-ask spread depends on $\mu$, it is difficult to express it as a function of $\mu$ explicitly. In order to understand the behavior of $\mu$, we give simple numerical examples in what follows. The figure below shows the relationship between $m$ and $\mu$ in the stable states in the case where $\overline{V} = 100$, $\underline{V} = 0$, $\alpha = 0.7,\ 0.8,\ 0.9$, $\epsilon^S = \epsilon^B = 0.05$, and $c = 1$. $\mu$ and the average value of the bid-ask spread decrease exponentially with the capacity of memory $m$. Thus, we show that the prices converge to the strong form efficient level of the price exponentially over time (as is shown in Easley and O'Hara (1992)) in the model in which the traders can choose whether they receive the private information or not.

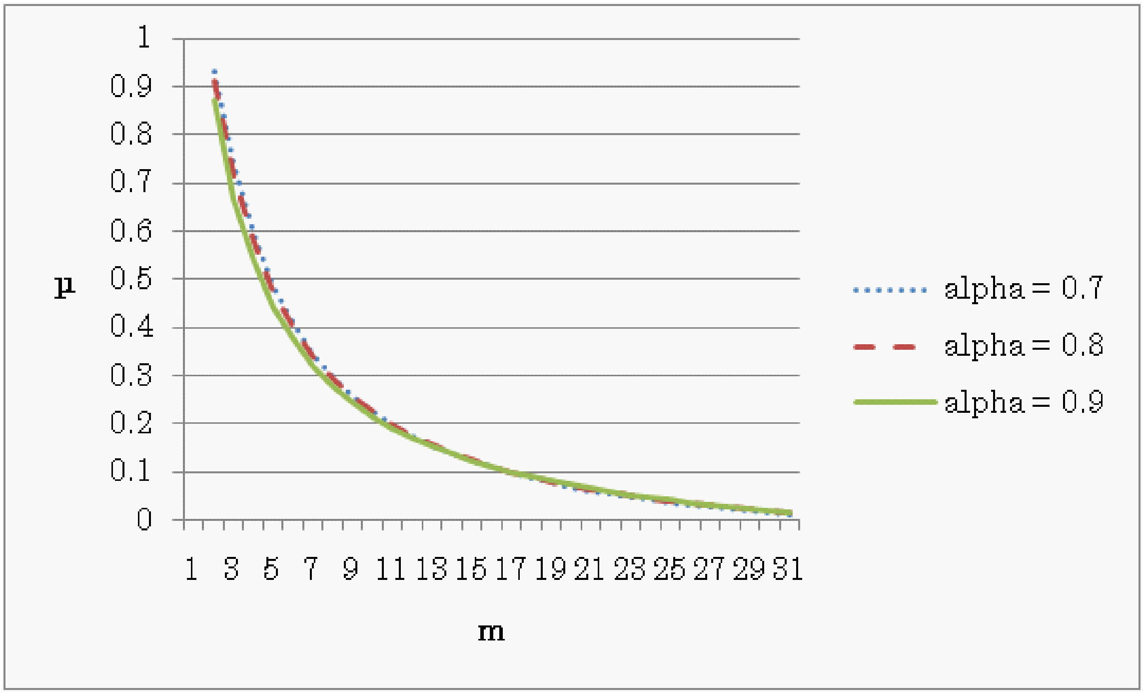


Figure 2

Intuitively, as $m$ becomes greater, the market maker can learn more from the past trade flow, and the asymmetric information is eased. He narrows down bid-ask spreads since the loss by the trade with informed traders is less. Since the information is conveyed through prices, the traders become less willing to receive the information. Even if they may have to trade for liquidity reasons, the loss is not large so much because the price is almost strong form efficient. The above result holds with various parameters.

The ration $\mu$ of information acquisition when $\alpha = 0.9$ is lower than when $\alpha = 0.7$, in case of memory $m$ is short. This might be against our intuition and debatable. For example, we can interpret in the following manner: If $\alpha$ is pretty close to 1, it is sure that traders can acquire the information almost certainly. However, the market maker knows this fact, and would set the bid-ask spread to a substantially wide level, corresponding to high $\mu$, so that the payoff of a informed trader comes to be small. That is, the benefit to get the information is relatively less than the cost, and the traders have a little incentive of information acquisition. When $m$ is longer, the market maker gives greater importance to the past trade on the decision making of the spread. The greater $\alpha$ is, the more the information acquisition is, which is consistent with our intuition.[3] Other comparative statics are intuitive, including that $\mu$ is uniformly greater if information acquisition cost $c$ is smaller.

# 5 Conclusion

We analyze the model in which the traders can choose either to receive private information or not when the capacity of memory of the market maker is limited. We obtain the result that the prices exponentially converge to the strong-form efficient level.

# References

Easley, D and M. O'Hara (1992) "Time and the Process of Security Price Adjustment" in *The Journal of Finance* **47**, 577-605.

[3] The similar with respect to $\epsilon^S \left(= \epsilon^B\right)$ occurs when $\epsilon^S$ is close to zero.